The Parameterized Complexity of Scheduling with Precedence Delays: Shuffle Product and Directed Bandwidth

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Plan

1. Motivation
   - Scheduling with precedence delays
   - The XNLP class
2. Contribution
   - Summary
   - Shuffle Product
   - Directed Bandwidth
3. Discussion
4. References

Outline

Cooking is scheduling

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Cooking = sequence of tasks ruled by a partial order and waiting time constraints.

Precedence delays

- If τ is a feasible schedule and $(i \xrightarrow{\ell_{i,j}} j)$:
  - Exact delay: $\boxed{i} \xrightarrow{=\ell_{i,j}} \boxed{j}$
  - Minimum delay: $\boxed{i} \xrightarrow{\geq \ell_{i,j}} \boxed{j}$
  - Maximum delay: $\boxed{i} \xrightarrow{\leq \ell_{i,j}} \boxed{j}$

Why precedence delays?

Powerful modeling tool

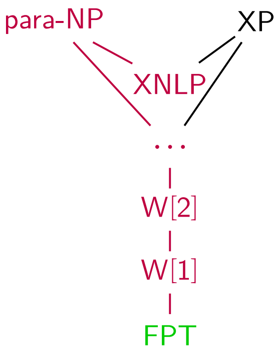
- Cooking.
- Setup times.
- Coupled tasks (e.g., loading / unloading operations).

Many related problems outside scheduling

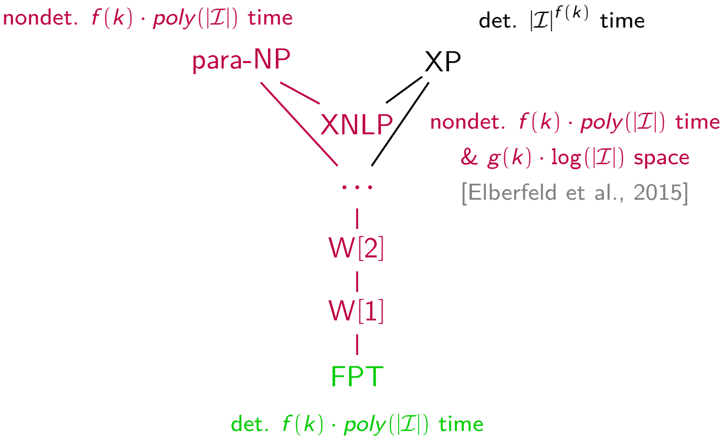
- Minimum: SHUFFLE PRODUCT.
  - W[2]-hard(#source words) [van Bevern et al., 2016].
- Exact: UNARY BIN PACKING.
  - W[1]-hard(#bins) [Jansen et al., 2013].
- Maximum: DIRECTED BANDWIDTH.
  - W[t]-hard(target value) for all t [Bodlaender, 2021].

Outline

Parameterized complexity classes



Parameterized complexity classes



Parameterized complexity classes

nondet. $f(k) \cdot poly(|\mathcal{I}|)$ time

para-NP

$\searrow$

XNLP — nondet. $f(k) \cdot poly(|\mathcal{I}|)$ time & $g(k) \cdot \log(|\mathcal{I}|)$ space [Elberfeld et al., 2015]

$\downarrow$

$\cdots$

$\downarrow$

W[2]

$\downarrow$

W[1]

Parameterized complexity classes

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XNLP

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[Elberfeld et al., 2015]

$\downarrow$

$\cdots$

$\downarrow$

W[2]

$\downarrow$

W[1]

Recent use of XNLP in scheduling

- [Bodlaender et al., 2022] Parallel identical machines, no precedence delays.
  - Width, #machines.
- [Mallem et al., 2026] Single machine with precedence delays.
  - Width, maximum delay value.

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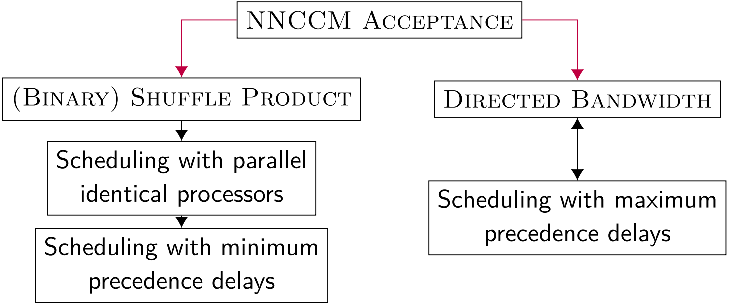
Summary of contributions

Problem	*Parameter*	*Result*
(Binary) Shuffle Product	#source words	XNLP-complete
Directed Bandwidth	target value plus width	XNLP-complete for directed acyclic graphs
	target value	XNLP-complete for directed trees



Nondeterministic Nonincreasing Counter Check Machine (NNCCM)

NNCCM Acceptance

Given: the description of a NNCCM:

- number of counters k, max. counter value n, number of checks r,
- a sequence of r checks in $([1, k] \times [0, n] \times [1, k] \times [0, n])^r$.

Parameter: number of counters k.

Question: is there an accepting run of the NNCCM?

Example: $k = 3$, $n = 2$, $r = 2$ checks: $((1, 0, 2, 0), (1, 1, 3, 0))$.

		r_1	r_2
c_1	0	1	1
c_2	0	0	0
c_3	0	0	1

		r_1	r_2
c_1	0	2	2
c_2	0	0	0
c_3	0	0	0

Outline

(Binary) Shuffle Product

(Binary) Shuffle Product

Given: source words $s_1, \ldots, s_k$ and target word t over a (binary) alphabet Σ.

Parameter: number of source words k.

Question: is t in the shuffle product of $s_1, \cdots, s_k$?

$$\begin{array}{rcccccccccc} s_1 & = & c & b & a & a \\ s_2 & = & a & b & c \\ s_3 & = & c & a \\ t & = & a & c & b & b & c & a & a & a & c \end{array}$$

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s_2 = & a & & & b & & & & & c \\
s_3 = & & & & & c & & & a & \\
t = & a & c & b & b & c & a & a & a & c
\end{array}$$

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s_3 = & & & & & c & & & a & \\
t = & a & c & b & b & c & a & a & a & c
\end{array}
$$

(Binary) Shuffle Product (cont.)

Example: $k = 3$, $n = 2$, $r = 2$ checks: $((1, 0, 2, 0), (1, 1, 3, 0))$.

		r_1	r_2
c_1	0	1	1
c_2	0	0	0
c_3	0	0	1

$$s_1 = a\ b\ \ \ a\ b\ a\ bb \quad a\ b\ a\ b\ a\ bba\ b \quad bba\ b$$
$$s_2 = a\ b\ a\ b\ a\ bba\ b \ (\dots)\ a\ b\ a\ b\ a\ b\ a\ b \ (\dots)\ b$$
$$s_3 = a\ b\ a\ b\ a\ b\ a\ b \quad a\ \ \ b\ a\ b\ a\ bb \quad b\ a\ bb$$
$$t = a^3\, b^5\, a^3\, b^5\, a^3\, b^4\, a^3\, b^5 \quad a^3\, b^5\, a^3\, b^5\, a^3\, b^4\, a^3\, b^5 \quad b^4\, a^3\, b^5\, a^3\, b^5$$

(Binary) Shuffle Product (cont.)

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\begin{array}{rcccccccccccccccccccccc}
s_1 = & a & b & & & a & \boxed{1} & a & bb & & a & b & a & b & a & \boxed{1} & a & b & & bb & a & b & & \\
s_2 = & a & b & a & b & a & \boxed{0} & a & b & (\ldots) & a & b & a & b & a & \boxed{0} & a & b & (\ldots) & b & & & & \\
s_3 = & a & b & a & b & a & \boxed{0} & a & b & & a & & & b & a & \boxed{1} & a & bb & & b & a & bb & & \\
t = & a^3 & b^5 & a^3 & b^5 & a^3 & b^4 & a^3 & b^5 & & a^3 & b^5 & a^3 & b^5 & a^3 & b^4 & a^3 & b^5 & & b^4 & a^3 & b^5 & a^3 & b^5
\end{array}
$$

Implication to scheduling problems

(BINARY) SHUFFLE PRODUCT

↓ [van Bevern et al., 2016]

Scheduling with parallel identical processors

↓ [Mallem et al., 2026]

Scheduling with minimum precedence delays

- Number of source words $\leftrightarrow$ width of precedence graph.
- Alphabet size $\leftrightarrow$ number of processors $\leftrightarrow$ max. delay value.

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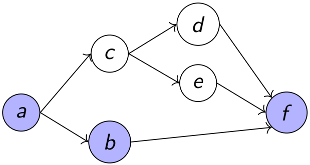
Directed Bandwidth

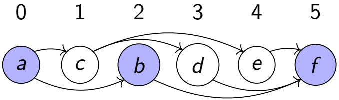
DIRECTED BANDWIDTH

Given: a directed acyclic graph $G = (V, A)$.

Parameter: target value B.

Question: is there $f : V \rightarrow \{0, \cdots, |V| - 1\}$ a bijective mapping such that, for each arc (i, j) in A, $f(j) - f(i) \leq B$?





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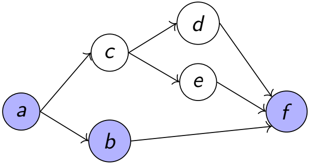

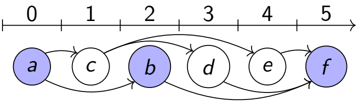

- $\Leftrightarrow$ single processor scheduling of unit-time tasks with equal-length **maximum precedence delays**.

Reduction from NNCCM Acceptance: directed acyclic graphs

Given: the description of a NNCCM:

- number of counters k, max. counter value n, number of checks r,
- a sequence of r checks in $([1,k] \times [0,n] \times [1,k] \times [0,n])^r$.

Parameter: number of counters k.

Key gadgets

- Floor path (length L).
  - Delay values forced to their maximum.
  - Other vertices come in "batches".
- For each of the k counters: a path of length $L + n$.
  - Two vertices in the same batch $\leftrightarrow$ counter increment.
- Check gadgets along the floor path and counter paths.
  - "Critical" batches at key steps of the floor path.
  - Cannot have two counter check gadgets in a critical batch.

(Binary) Shuffle Product (recall)

Example: $k = 3,\ n = 2,\ r = 2$ checks: $((1, 0, 2, 0),\ (1, 1, 3, 0))$.

$$
\begin{array}{c|c|c|c|}
 & & r_1 & r_2 \\
\hline
c_1 & 0 & 1 & 1 \\
\hline
c_2 & 0 & 0 & 0 \\
\hline
c_3 & 0 & 0 & 1 \\
\hline
\end{array}
$$

$$
\begin{array}{rcccccccccccccccccccccccc}
s_1 = & a & b & & & a & \boxed{b} & a & bb & & a & b & a & b & a & \boxed{bb} & a & b & & bb & a & b & & \\
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Reduction from NNCCM Acceptance: directed trees

Given: the description of a NNCCM:

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- a sequence of r checks in $([1, k] \times [0, n] \times [1, k] \times [0, n])^r$.

Parameter: number of counters k.

Key ideas

- Start from previous construction.
- Adapt counter gadgets.
  - Width becomes unbounded.
- Deal with added flexibility.
  - "Tail" gadgets to fix the order of path-ending vertices.

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Discussion: scheduling with precedence delays

Problem	*Parameter*	*Result*
(BINARY) SHUFFLE PRODUCT	#source words	XNLP-complete
DIRECTED BANDWIDTH	target value plus width	XNLP-complete for directed acyclic graphs
	target value	XNLP-complete for directed trees

Takeaway

- Managing a restaurant is (essentially) XNLP-complete.
- Scheduling as a complexity hub.

Open questions

- UNARY BIN PACKING with a bounded number of bins.
  - ↔ scheduling unit-time tasks with equal-length **exact delays**.
  - W[1]-hard [Jansen et al., 2013], in XP [Garey and Johnson, 1979].
- Minimum delays, constant max. delay value.
  - ↔ no delays, constant #processors (open since [Ullman, 1975]).
  - XNLP-hard [Bodlaender et al., 2022].

Discussion: scheduling with precedence delays

Problem	*Parameter*	*Result*
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