Parameterized Complexity in Scheduling

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Plan

1. Base concept
   - Motivation
   - Fixed-parameter tractability (FPT)
   - Parameterized hardness
2. Recent developments
   - The XNLP class
   - Kernelization
   - N-fold integer programs
3. Conclusion
4. References

Outline

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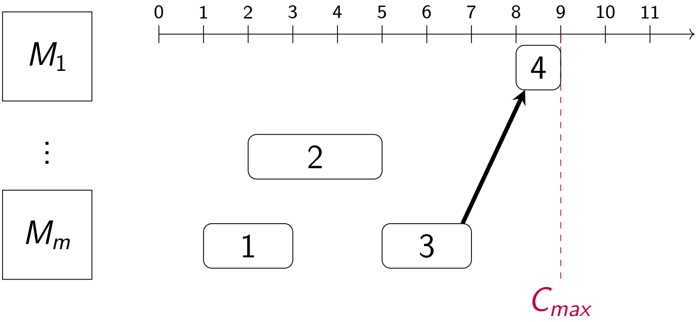
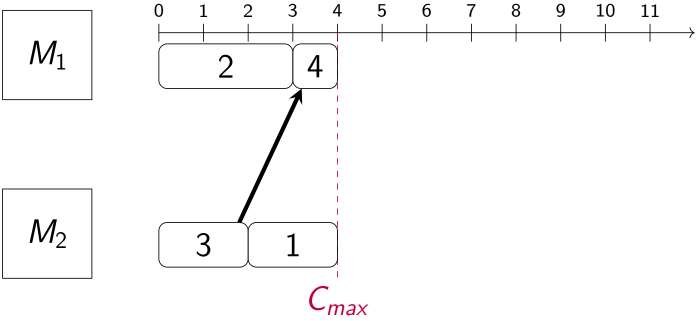
How hard is scheduling?



Set of tasks $\mathcal{T} = \{1, \cdots, n\}$.
Number of machines $m \in \mathbb{N}$.
Precedence graph $G = (\mathcal{T}, A)$.

Goal: schedule $\tau : \mathcal{T} \mapsto \mathbb{Z}^+$ minimizing makespan C_{max}.



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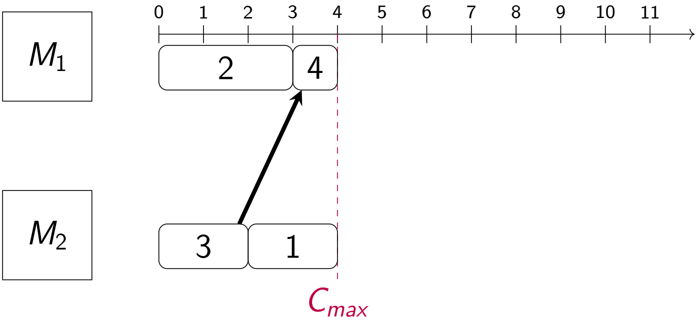

Set of tasks $\mathcal{T} = \{1, \cdots, n\}$.
Number of machines $m \in \mathbb{N}$.
Precedence graph $G = (\mathcal{T}, A)$.

Strongly NP-complete [Garey and Johnson, 1978].

Goal: schedule $\tau : \mathcal{T} \mapsto \mathbb{Z}^+$ minimizing makespan C_{max}.

Strongly NP-complete even with unit-time tasks or empty G.

Outline

Going beyond NP-hardness

- Problem $\mathcal{P}$, instance $\mathcal{I}$.
  - Parameter $\bar{k} : \mathcal{P} \to \mathbb{N}$.
  - Instance $\mathcal{I}$ with parameter value $\bar{k}(\mathcal{I}) \leq k$.
- P: deterministic time $poly(|\mathcal{I}|)$.
  - FPT: deterministic time $f(k) \cdot poly(|\mathcal{I}|)$.
  - XP: deterministic time $|\mathcal{I}|^{f(k)}$.

Fixed-parameter algorithms in scheduling

- MIP with a bounded number of integer variables.
  - Initiated by [Mnich and Wiese, 2015].
  - Meta-theorems: MILP [Lenstra, 1983], MICP [Dadush et al., 2011].
  - Bounded number of different task lengths [Hermelin et al., 2021].
- Dynamic Programming
  - Width of precedence graph [van Bevern et al., 2016].
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- P: deterministic time $poly(|\mathcal{I}|)$.
  - FPT: deterministic $f(k) \cdot poly(|\mathcal{I}|)$ time.
- NP: nondeterministic time $poly(|\mathcal{I}|)$.
  - para-NP: nondeterministic $f(k) \cdot poly(|\mathcal{I}|)$ time.
  - para-NP-complete: NP-complete for some fixed k.

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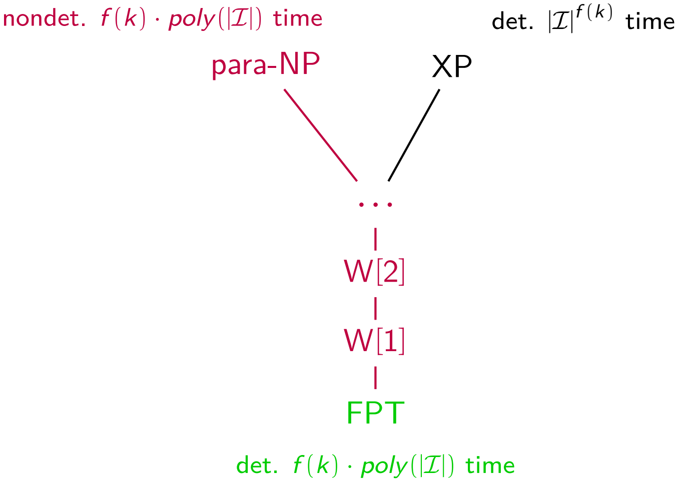
nondet. $f(k) \cdot poly(|\mathcal{I}|)$ time
det. $|\mathcal{I}|^{f(k)}$ time
para-NP
XP
...
W[2]
W[1]
FPT
det. $f(k) \cdot poly(|\mathcal{I}|)$ time

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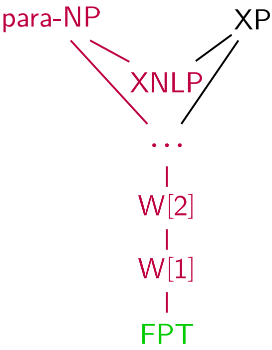

The XNLP class

nondet. $f(k) \cdot poly(|\mathcal{I}|)$ time

para-NP

det. $|\mathcal{I}|^{f(k)}$ time

XP

XNLP

nondet. $f(k) \cdot poly(|\mathcal{I}|)$ time
& $g(k) \cdot \log(|\mathcal{I}|)$ space
[Elberfeld et al., 2015]

$\cdots$

W[2]

W[1]

FPT

det. $f(k) \cdot poly(|\mathcal{I}|)$ time

The XNLP class

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$$\text{para-NP} \to \text{XNLP} \to \cdots \to \text{W}[2] \to \text{W}[1]$$

XNLP: nondet. $f(k) \cdot poly(|\mathcal{I}|)$ time & $g(k) \cdot \log(|\mathcal{I}|)$ space [Elberfeld et al., 2015]

The XNLP class

nondet. $f(k) \cdot poly(|\mathcal{I}|)$ time

para-NP

$\searrow$

XNLP — nondet. $f(k) \cdot poly(|\mathcal{I}|)$ time & $g(k) \cdot \log(|\mathcal{I}|)$ space [Elberfeld et al., 2015]

$\downarrow$

$\cdots$

$\downarrow$

W[2]

$\downarrow$

W[1]

Recent use of XNLP in scheduling

- Parallel identical machines.
  - $P|prec, p_j = 1|C_{max}$ is XNLP-hard parameterized by the number of machines [Bodlaender et al., 2022].
- Single machine with precedence delays.
  - [Mallem et al., 2026, Bodlaender and Mallem, 2026].

Outline

Kernelization

- Decision problem $\mathcal{P}$, instance $\mathcal{I}$.
  - Parameter $\bar{k} : \mathcal{P} \rightarrow \mathbb{N}$.
  - Instance $\mathcal{I}$ with parameter value $\bar{k}(\mathcal{I}) \leq k$.
- Kernelization algorithm: outputs $\mathcal{I}'$ in time $poly(|\mathcal{I}|)$ such that:
  - $\mathcal{I}'$ is equivalent to $\mathcal{I}$.
  - $|\mathcal{I}'| \leq f(k)$ for some computable function f.
  - $\bar{k}(\mathcal{I}') \leq g(k)$ for some computable function g.

Reduction rules in scheduling

- Numerical values (time, weight, etc.).
  - Meta-theorem [Frank and Tardos, 1987].
  - Knapsack [Etscheid et al., 2017, Gurski et al., 2019].
- Number of tasks.
  - Single machine with task time windows [Mallem et al., 2024].

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N-fold integer programming

$A\mathbf{x} \leq \mathbf{b}$ with A taking the following form:

$$\begin{pmatrix} A_1^1 & A_1^2 & \cdots & A_1^N \\ A_2^1 & 0 & \cdots & 0 \\ 0 & A_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_2^N \end{pmatrix}$$

Highlights

- MoMA 2025 workshop (GT CoA, Graphes, AGaPe, GOThA).
  - Invited lecture: *Integer Programming meets Fixed-Parameter Tractability* by Alexandra Lassota (TU Eindhoven).
- High-multiplicity packing/scheduling [Knop et al., 2019].
  - General framework with a wide range of suitable objective functions.

Outline

Conclusion

Takeaway

- A toolkit to better understand scheduling problems.
- More efficient exact (and approximation!) algorithms.

Going further

- Implementation and experiments.
- Fine-grained complexity.
  - See [Swennenhuis, 2022] and Klaus Jansen's work (Kiel University).
- Scheduling Zoo.
  - https://schedulingzoo.lip6.fr ("advanced" interface).

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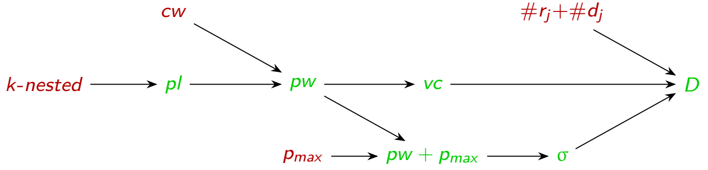
Parameter hierarchy on $1|r_j, d_j|C_{max}$



k-nested: depth of the forest inferred by the "strictly included on both ends" relation.

cw: clique-width of the time window interval graph.

p_{max}: maximum processing time.

$\#r_j + \#d_j$: number of distinct release date and deadline values.

pl: proper level.

pw: pathwidth of the time window intersection graph.

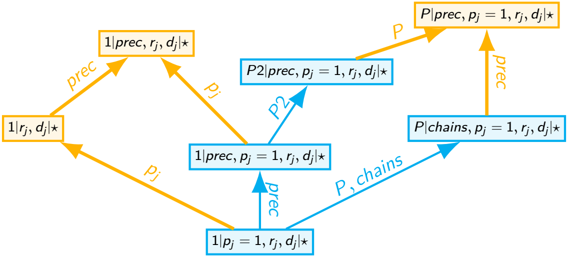
vc: vertex cover number of the time window intersection graph.

σ slack: $\max_j(d_j - r_j - p_j)$.

D: maximum deadline.

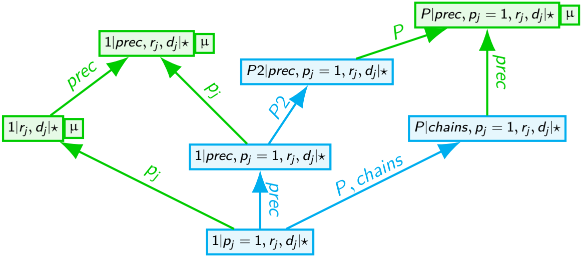
- $a \longrightarrow b$ if and only if, for some computable function f, $a(\mathcal{I}) \leq f(b(\mathcal{I}))$ for every $\mathcal{I} \in \mathcal{P}$.

Parameter μ



$P|prec, p_j = 1, r_j, d_j|\star$
$1|prec, r_j, d_j|\star$
$P2|prec, p_j = 1, r_j, d_j|\star$
$1|r_j, d_j|\star$
$1|prec, p_j = 1, r_j, d_j|\star$
$P|chains, p_j = 1, r_j, d_j|\star$
$1|p_j = 1, r_j, d_j|\star$
prec
p_j
P
P2
prec
p_j
prec
P, chains

Parameter μ



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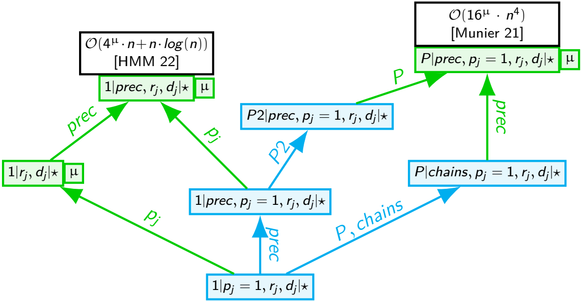
$\mathcal{O}(4^{\mu} \cdot n + n \cdot log(n))$
[HMM 22]
$1|prec, r_j, d_j|\star$
μ
$\mathcal{O}(16^{\mu} \cdot n^4)$
[Munier 21]
$P|prec, p_j = 1, r_j, d_j|\star$
μ
P
$P2|prec, p_j = 1, r_j, d_j|\star$
prec
prec
pj
P2
$1|r_j, d_j|\star$
μ
$P|chains, p_j = 1, r_j, d_j|\star$
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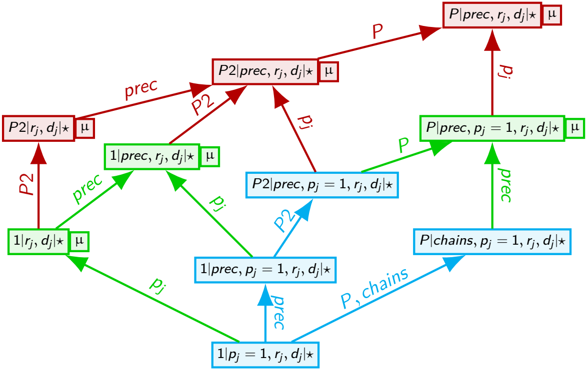

$P|prec, r_j, d_j|\star$ μ
P
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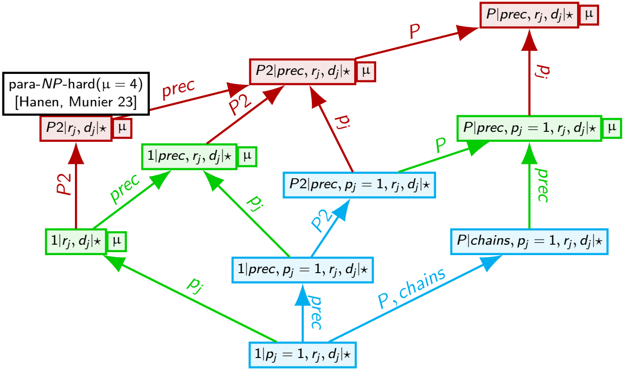
$P|prec, r_j, d_j|\star$ μ
P
$P2|prec, r_j, d_j|\star$ μ
pj
para-NP-hard(μ = 4)
[Hanen, Munier 23]
prec
P2
$P2|r_j, d_j|\star$ μ
pj
$P|prec, p_j = 1, r_j, d_j|\star$ μ
$1|prec, r_j, d_j|\star$ μ
P
$P2|prec, p_j = 1, r_j, d_j|\star$
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pj
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P2
$1|r_j, d_j|\star$ μ
$P|chains, p_j = 1, r_j, d_j|\star$
$1|prec, p_j = 1, r_j, d_j|\star$
pj
P , chains
prec
$1|p_j = 1, r_j, d_j|\star$