Graph parameters for single machine scheduling with time windows

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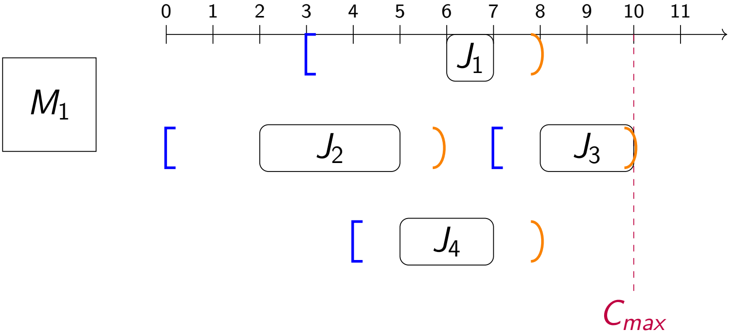
1 April 2026

Outline

Outline

Introduction to scheduling

- Introduction
- Vertex cover number
- Pathwidth
- Proper level
- Conclusion
- References



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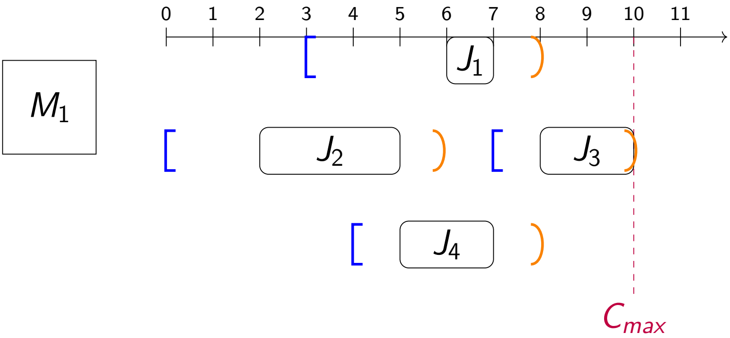

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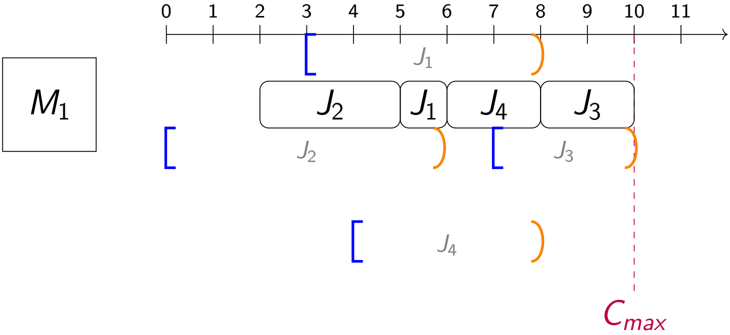

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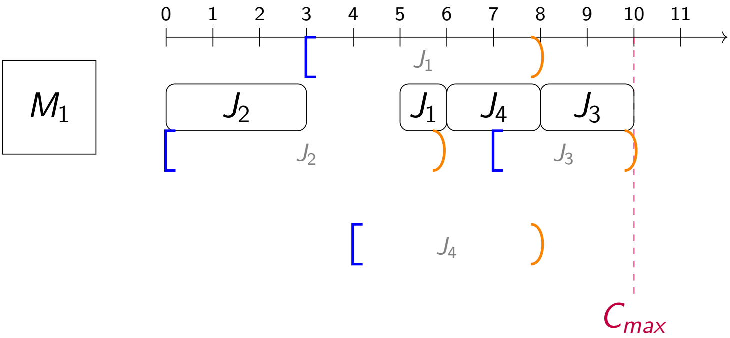

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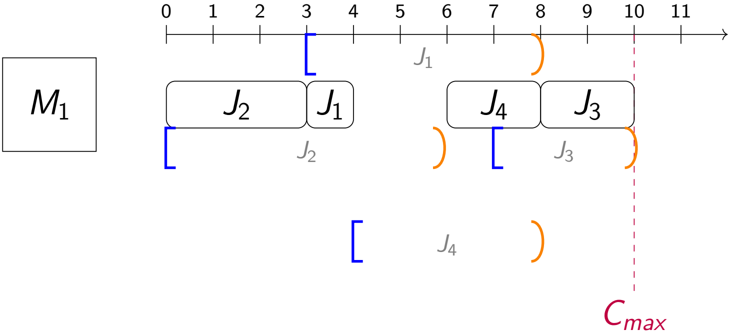

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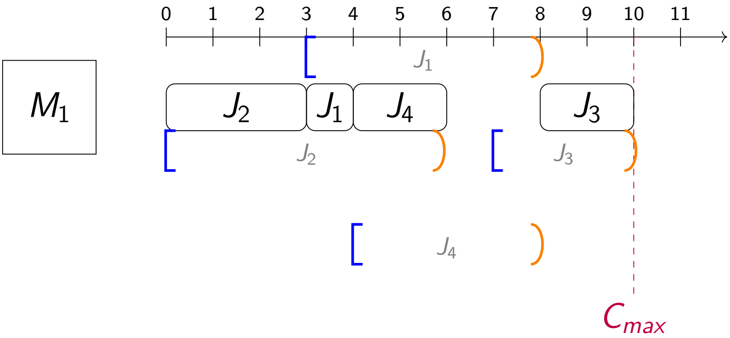

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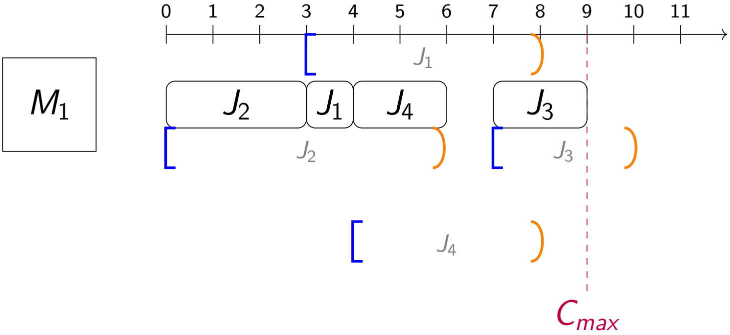

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Dynamic programming state (k, S_k)

- k = number of completed jobs.
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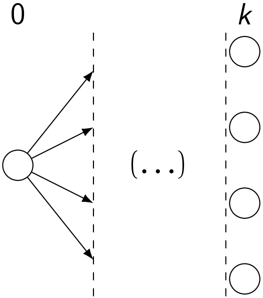

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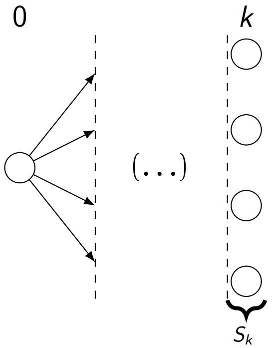

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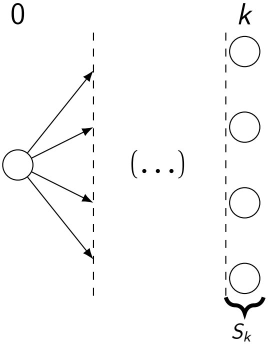

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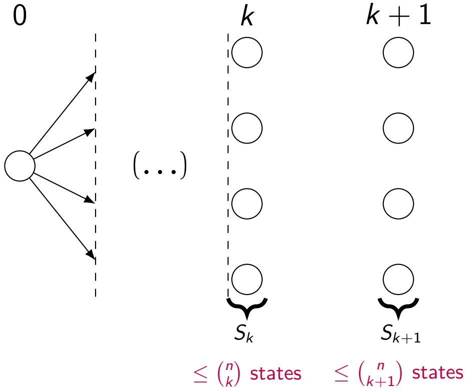

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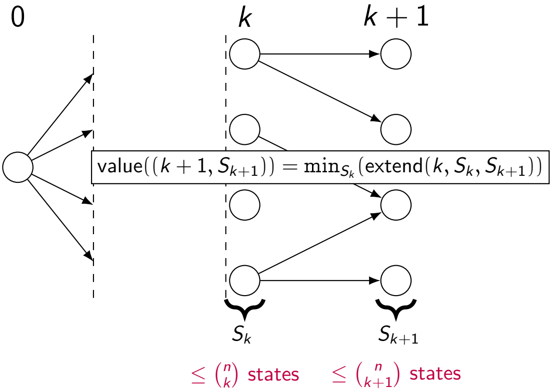

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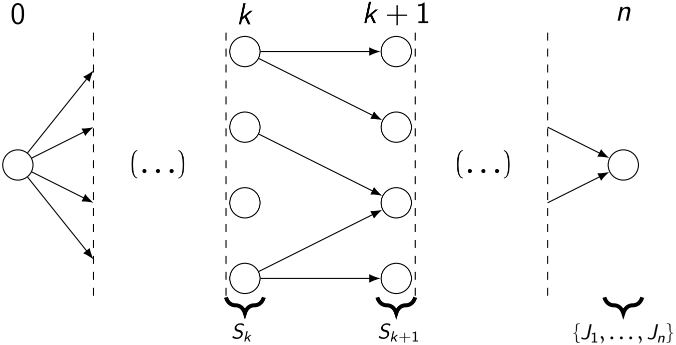

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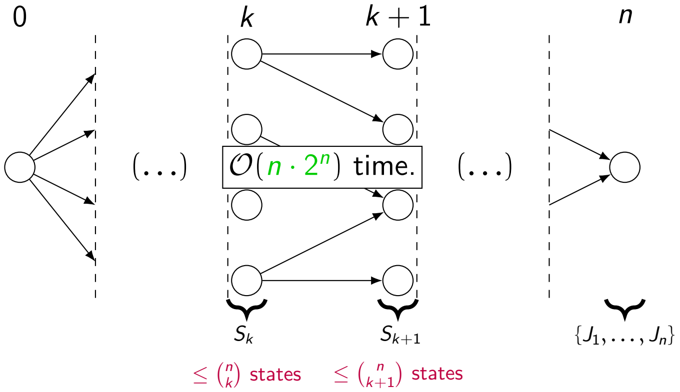

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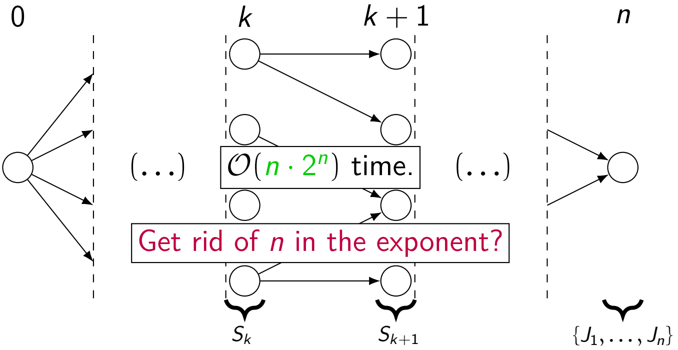

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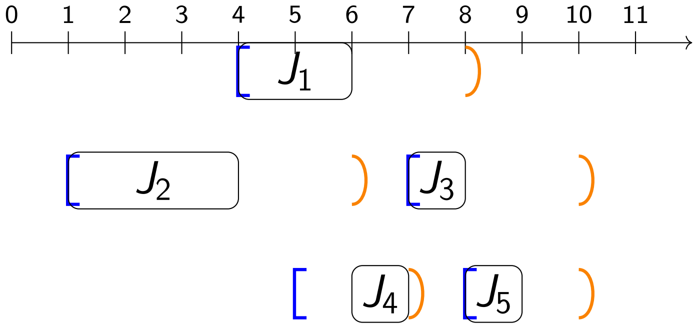
- **Example:** $1|r_j, d_j|C_{max}$ parameterized by n is in FPT.
  - (Note: input size in $\mathcal{O}(n \cdot \log(d_{\max}))$.)

Outline

Vertex cover number *vc*



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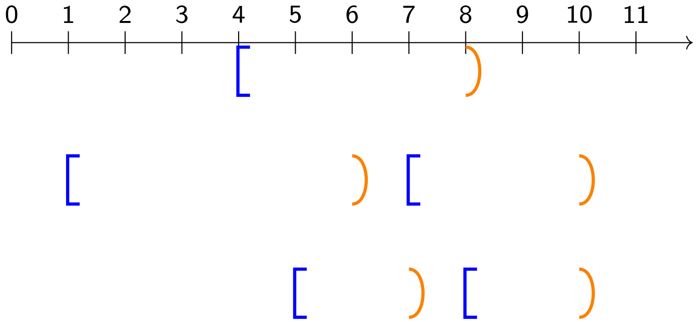

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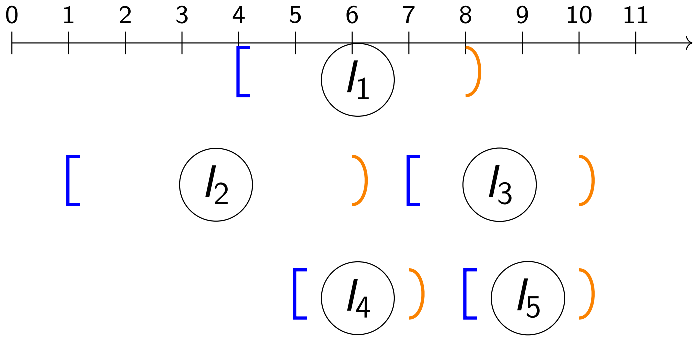

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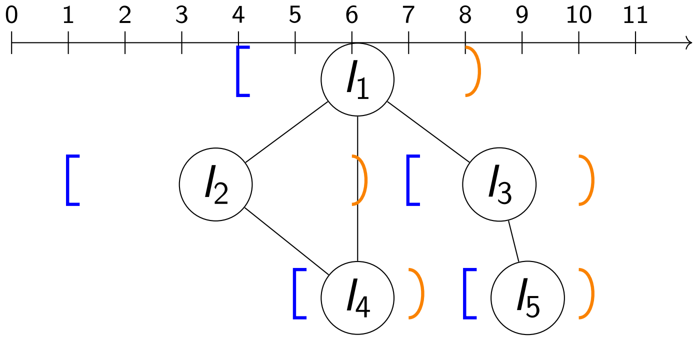

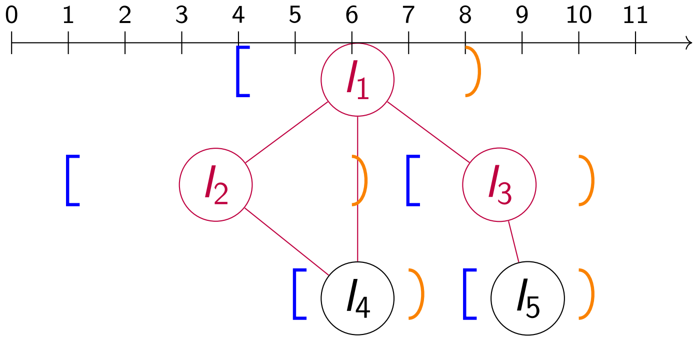
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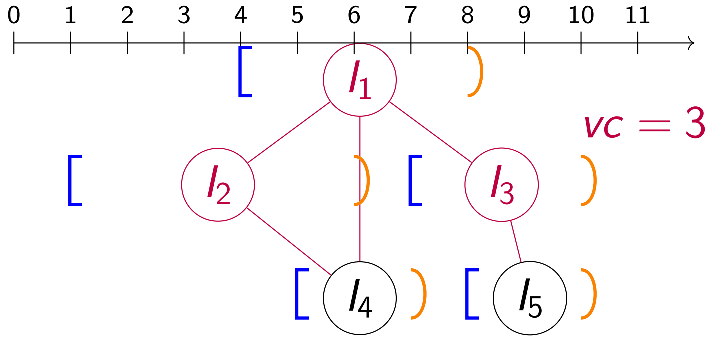

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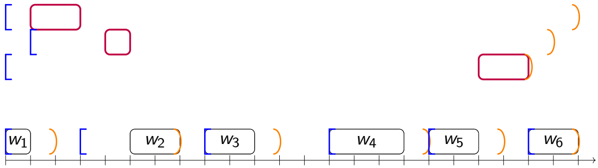

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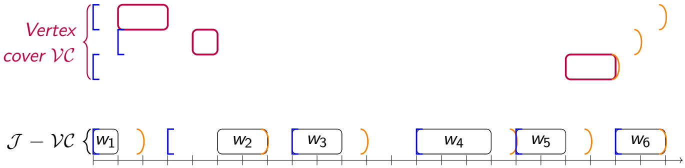

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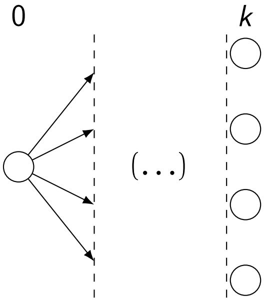

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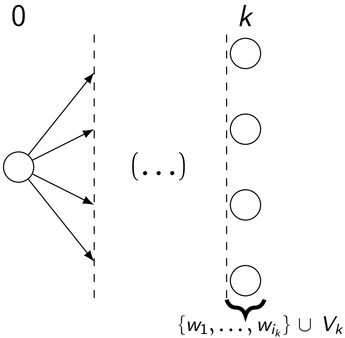

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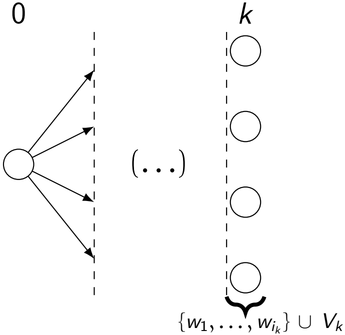

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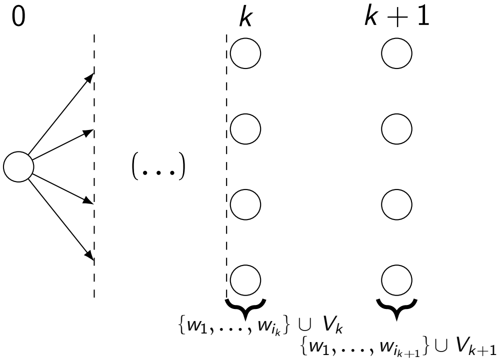

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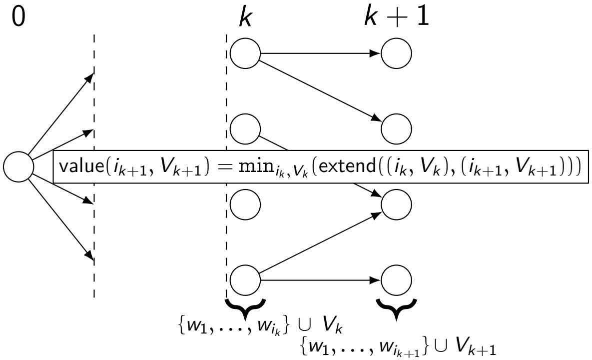

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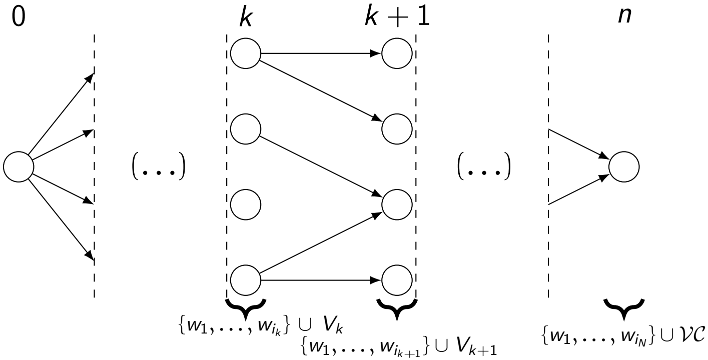

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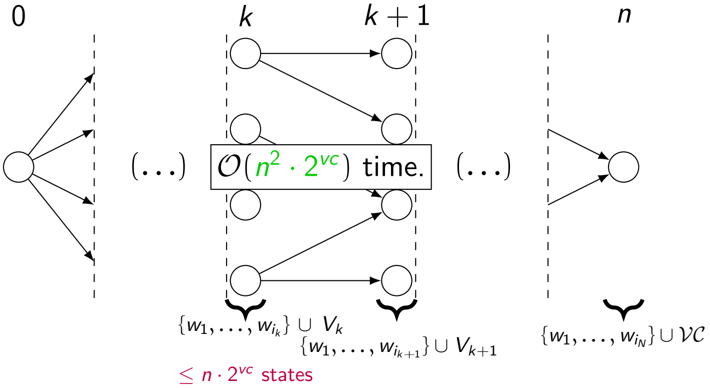

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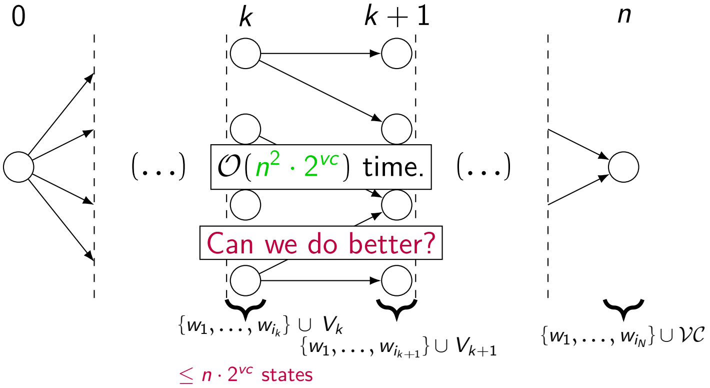

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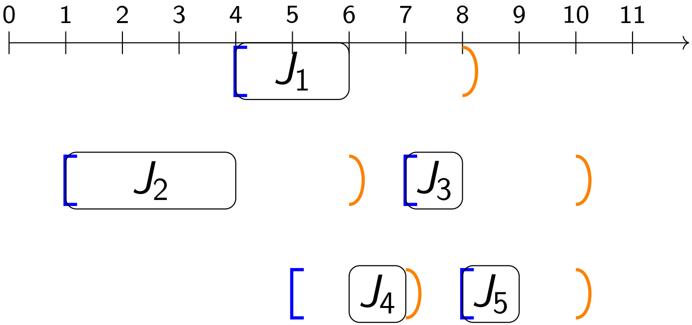

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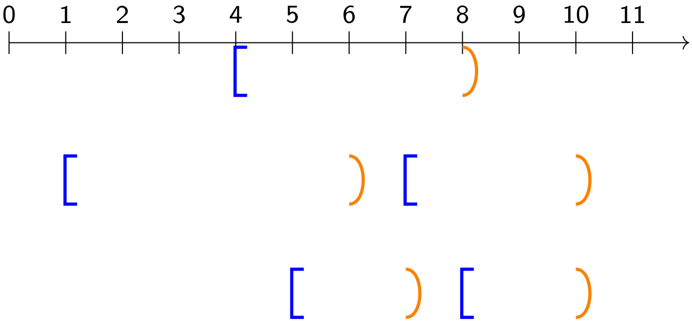

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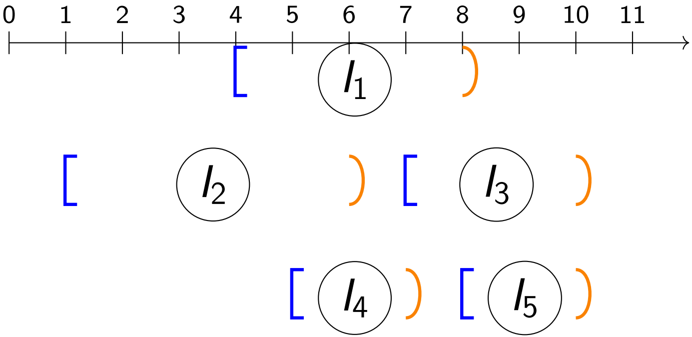

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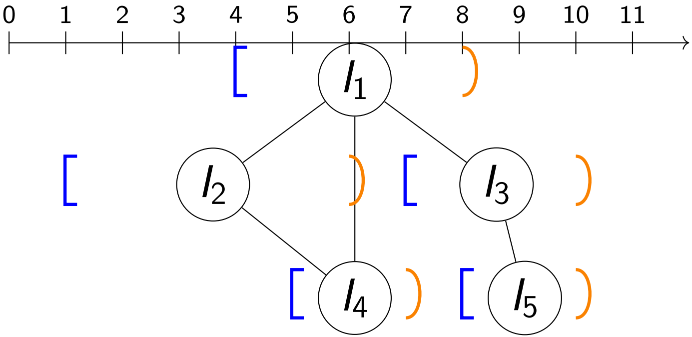

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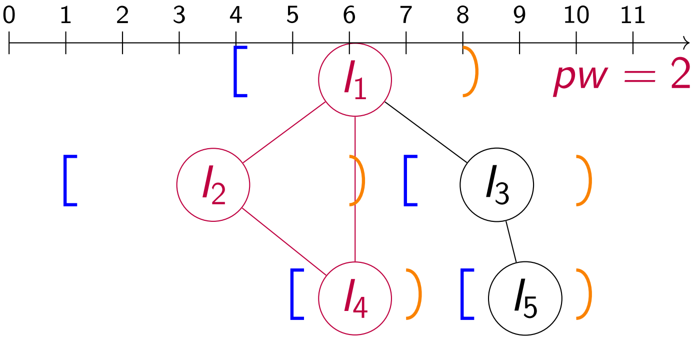

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$$\omega(G) = \chi(G) = tw(G) + 1 = pw(G) + 1.$$

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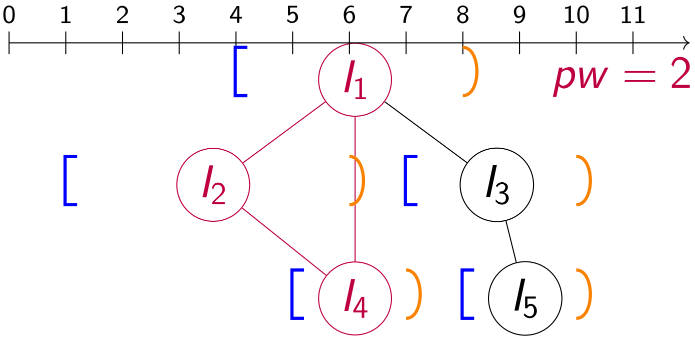

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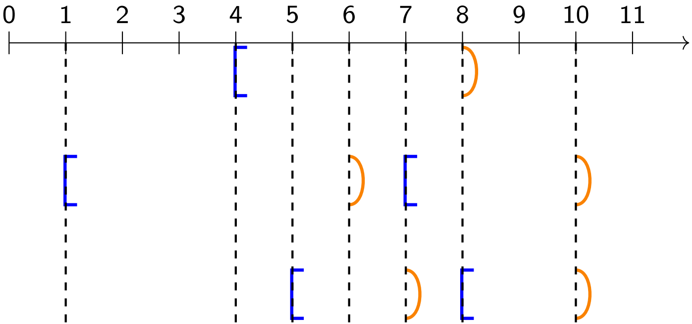

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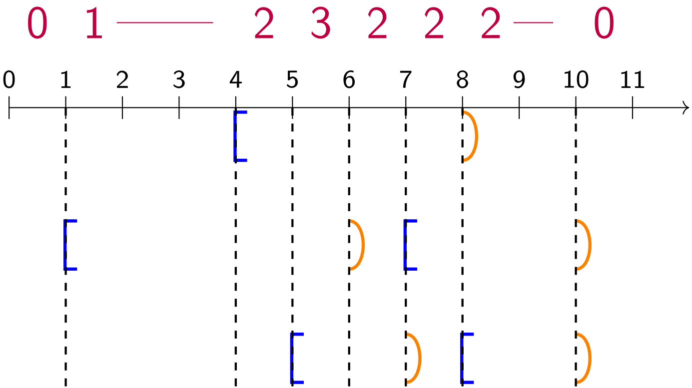

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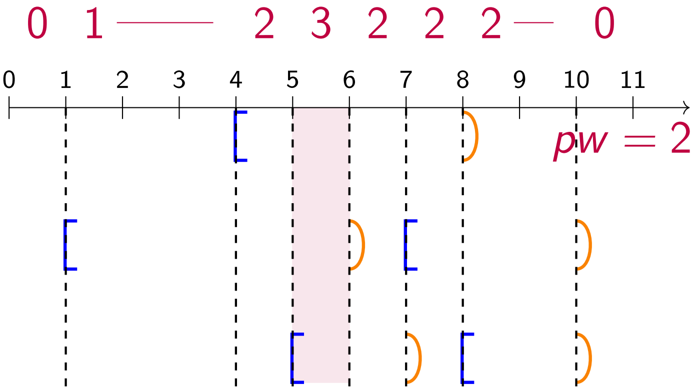

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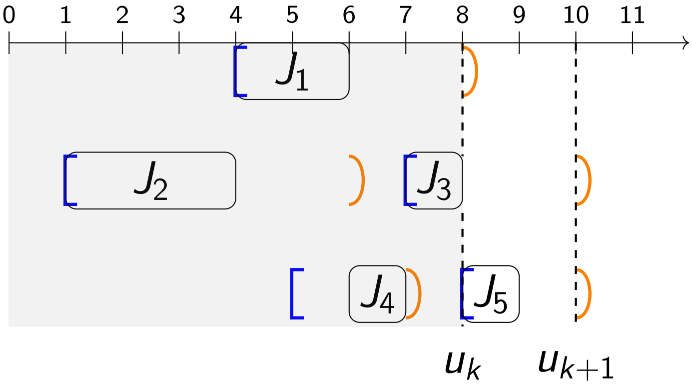

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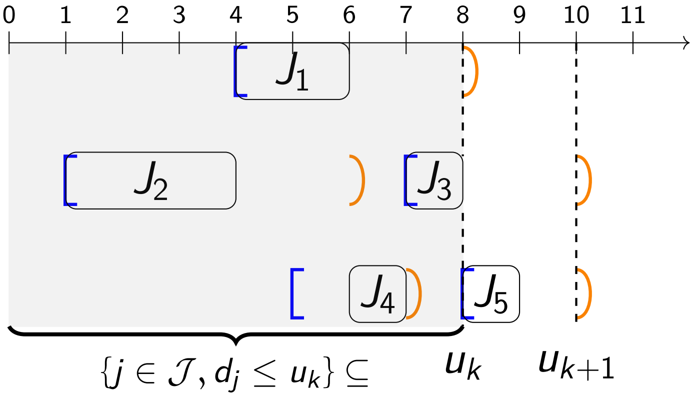

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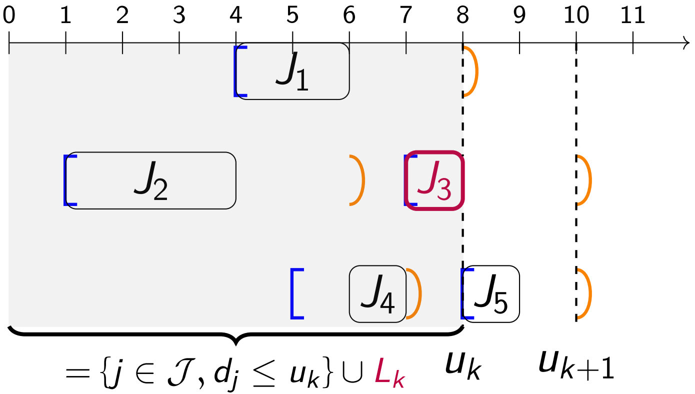

- $J_j \in L_k \implies [u_k, u_{k+1}) \subseteq [r_j, d_j)$: at most $pw + 1$ such jobs.

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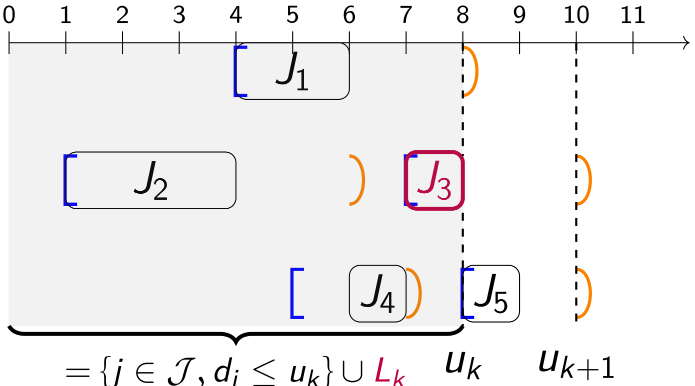

$$= \underbrace{\{j \in \mathcal{J}, d_j \leq u_k\}}_{= Z_k} \cup L_k$$

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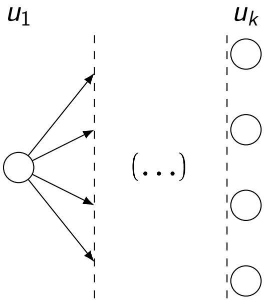

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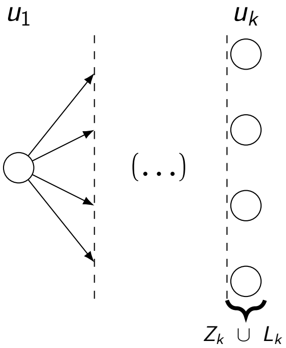

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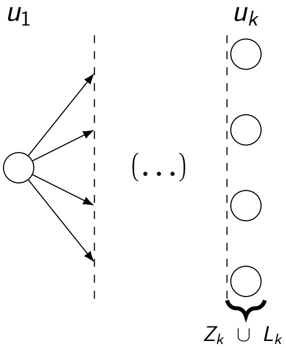

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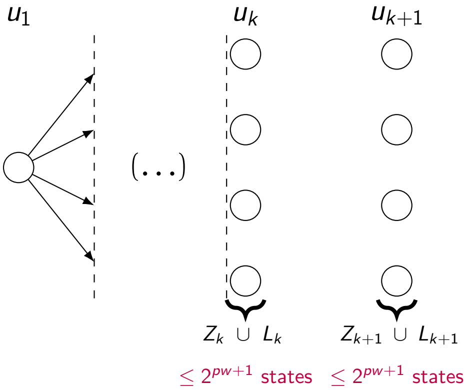

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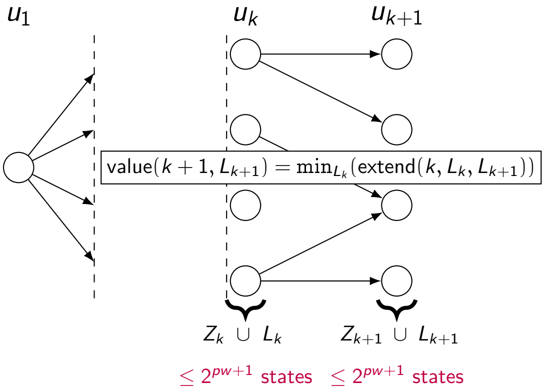

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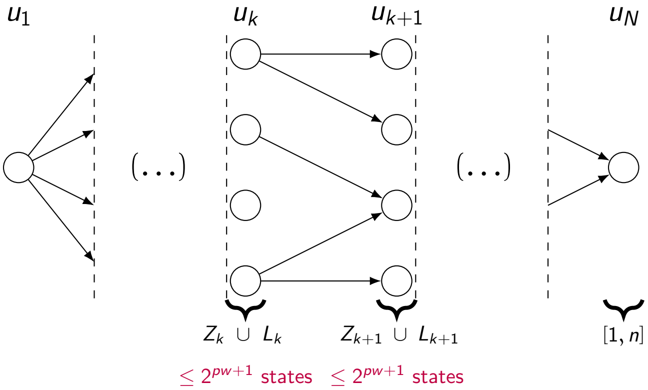

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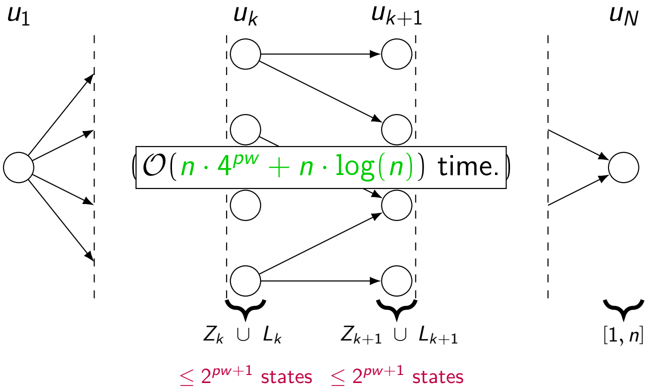

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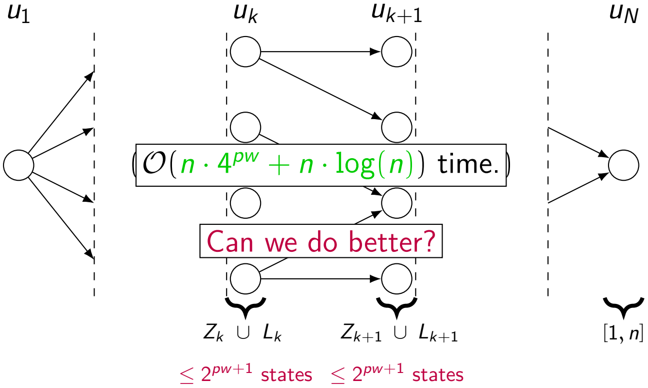
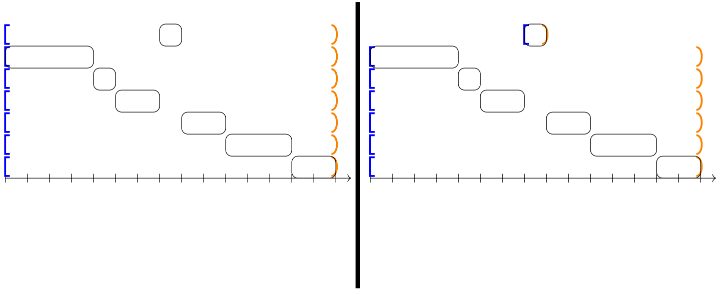


Outline

Proper level

- Two single machine instances with time windows.



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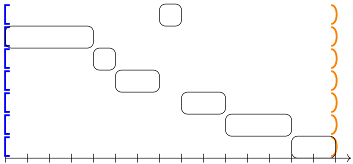

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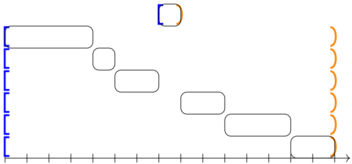
Interval graph: K_n



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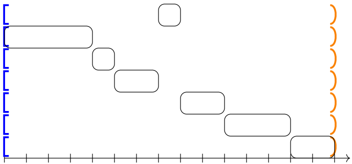

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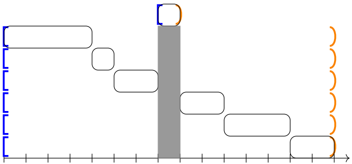

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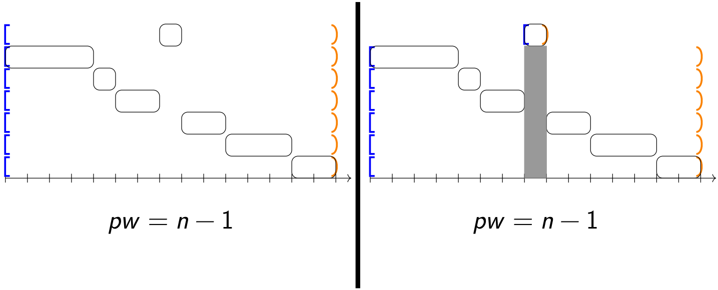

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Definition: Proper sets and proper level

- $\forall J_i \in \mathcal{J}, \Pi_i = \{J_j \in \mathcal{J}, r_j < r_i \wedge d_i < d_j\}$
- $pl = max_{J_i \in \mathcal{J}} \; |\Pi_i|$

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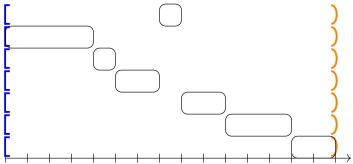

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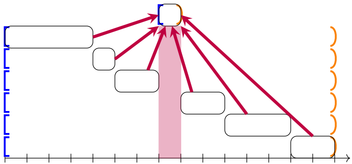

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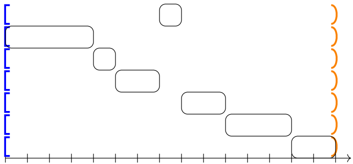

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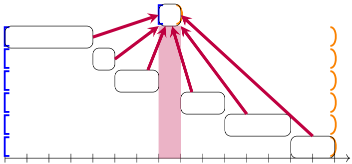

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- $pl = max_{J_i \in \mathcal{J}} |\Pi_i|$ [Proskurowski and Telle, 1999].

The "Earliest Deadline first" rule

- "$\prec$" = lexicographic order (nondecr. d_j, nondecr. r_j) on $\mathcal{J}$.
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[Lawler and Moore, 1969] "Earliest Deadline first" rule (ED)

Schedule τ follows the (ED) rule if and only if:

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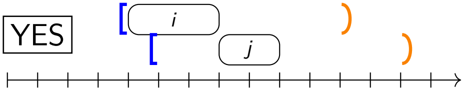

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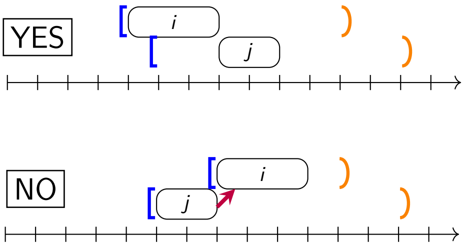

Relaxing the “Earliest Deadline first” rule

Definition: “weak Earliest Deadline first” rule (wED)

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$\forall (i,j) \in \mathcal{J}^2, i \prec j \implies$

(1) $\tau(i) < \tau(j)$, or

(2) $j \in \Pi_i$, or

(3) $\exists h \prec i, \tau(j) < \tau(h) < \tau(i)$.

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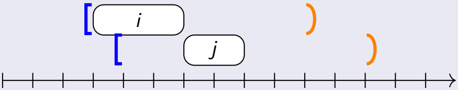

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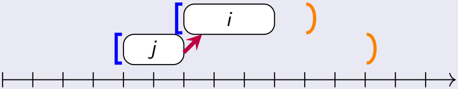

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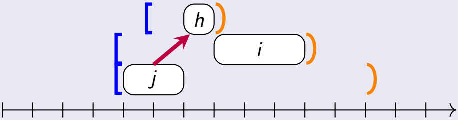

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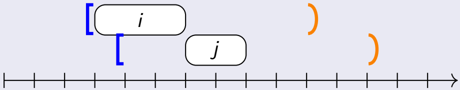

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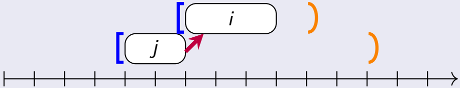

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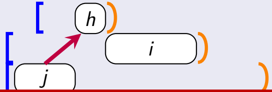

Proposition

On $1|r_j, d_j|C_{max}$ the (wED)-following schedules are dominant.

Definition: Summit

A summit is a job i in $\mathcal{J}$ such that: $\forall j \in \mathcal{J}, i \notin \Pi_j$.

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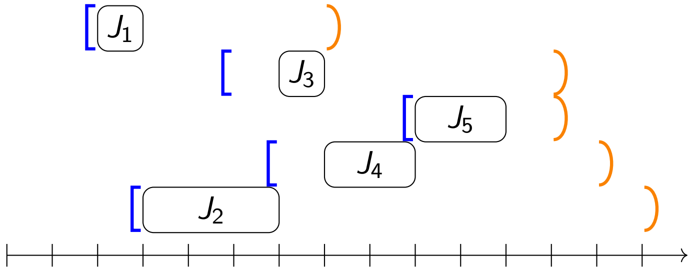

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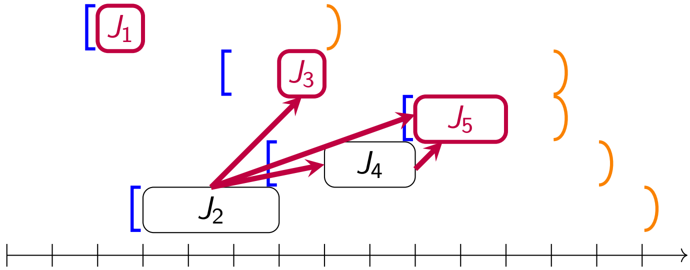

Lemma

Let $s_1 \prec \ldots \prec s_N$ be the summits of the instance ($N \leq n$).
If a schedule τ follows the (wED) rule, then:

(i) τ is of the form $T_1 s_1 \ldots s_k T_{k+1} s_{k+1} \ldots s_N T_{N+1}$,

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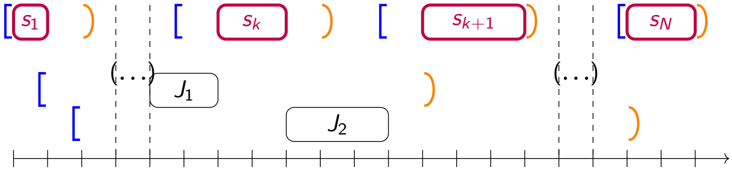

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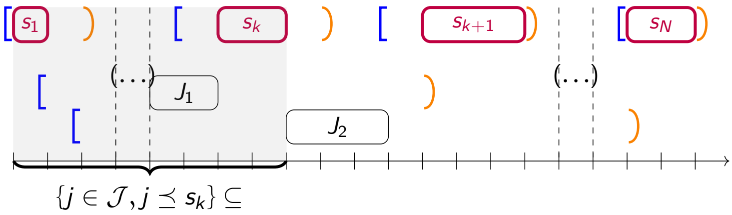

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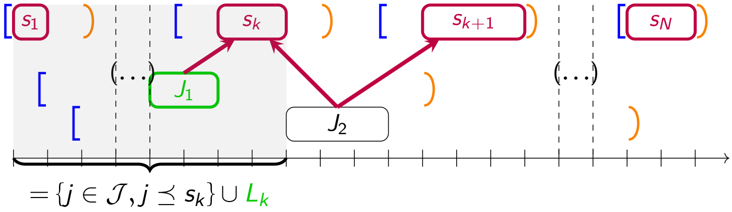

- $j \in L_k \implies j \in \Pi_{s_k}$: at most *pl* such jobs j.

Dynamic programming with proper level pl

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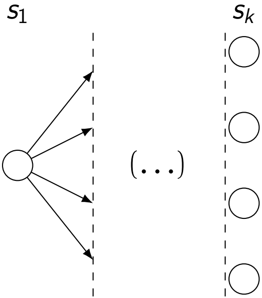

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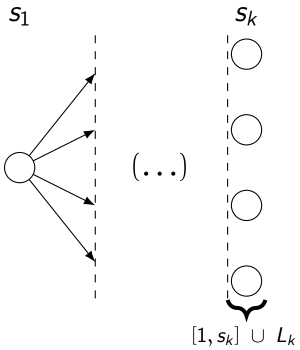

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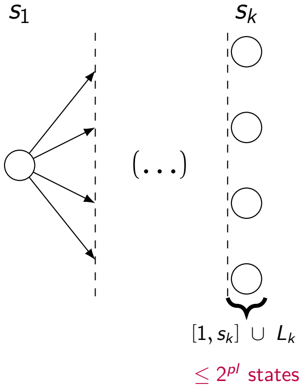

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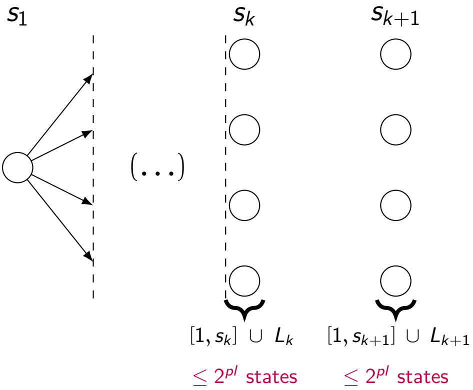

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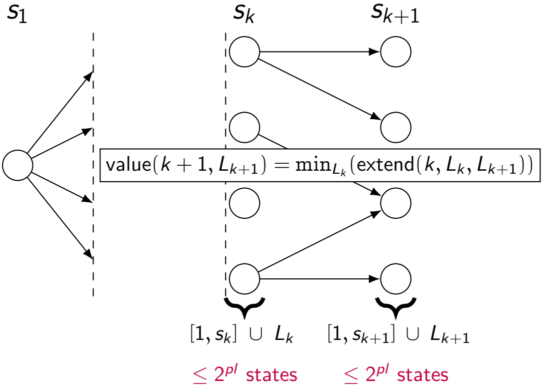

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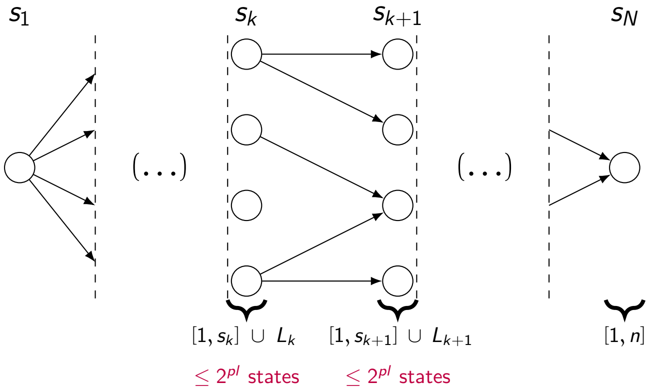

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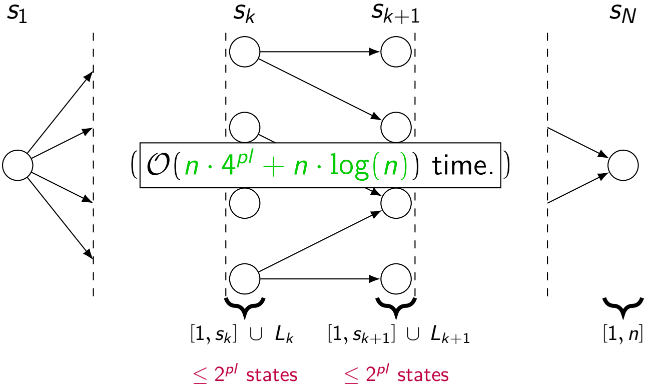

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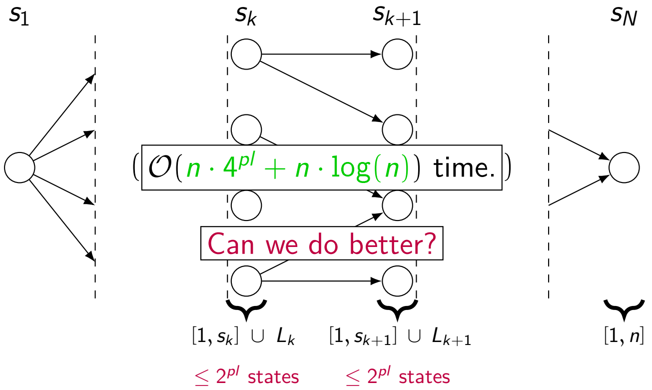

Outline

1. Introduction
2. Vertex cover number
3. Pathwidth
4. Proper level
5. Conclusion
6. References

Takeaway

- Better understanding of scheduling problems.
- More efficient exact algorithms.

Conclusion

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Going further

- $4^{pl} \longrightarrow 2^{pl}$?
- Lower bound on $f(k)$: **fine-grained complexity**.
- Poly-time compression to $g(k)$ size: **kernelization**.
- Parameterized hardness: para-NP, W-hierarchy, XNLP.

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Recent news

- IWOCA (with Bodlaender): scheduling with precedence delays.
- FUN!

Parameter hierarchy on $1|r_j, d_j|\star$

$$pl \longrightarrow pw \longrightarrow vc$$

Parameter hierarchy on $1|r_j, d_j|\star$

$$
\begin{tikzcd}
 & cw \arrow[dr] & & & \#r_j+\#d_j \arrow[dr] & \\
k\text{-}nested \arrow[r] & pl \arrow[r] & pw \arrow[r] \arrow[dr] & vc \arrow[rr] & & D \\
 & & p_{max} \arrow[r] & pw + p_{max} \arrow[r] & \sigma \arrow[ur] &
\end{tikzcd}
$$

k-nested: depth of the forest inferred by the "strictly included on both ends" relation.

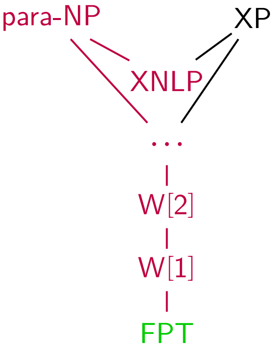
cw: clique-width of the time window interval graph.

p_{max}: maximum processing time.

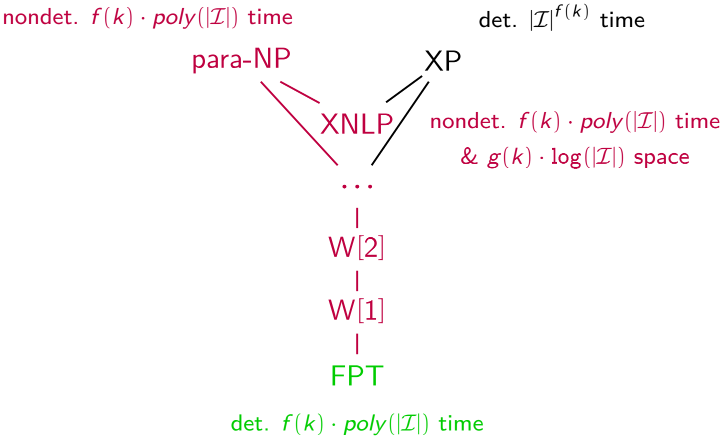
$\#r_j+\#d_j$: number of distinct release date and deadline values.

σ slack: $\max_j(d_j - r_j - p_j)$.

D: maximum deadline.


para-NP
XP
XNLP
...
W[2]
W[1]
FPT

Parameterized complexity classes



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