Parameterized Complexity of Parallel Machine Scheduling with Time Windows

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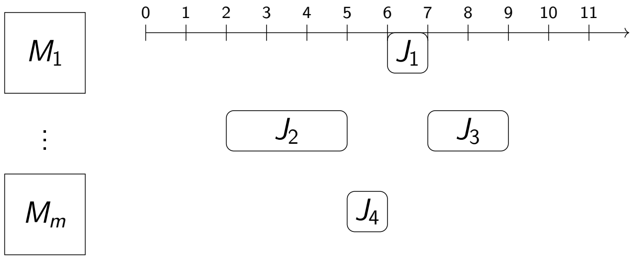
16 May 2025

Outline

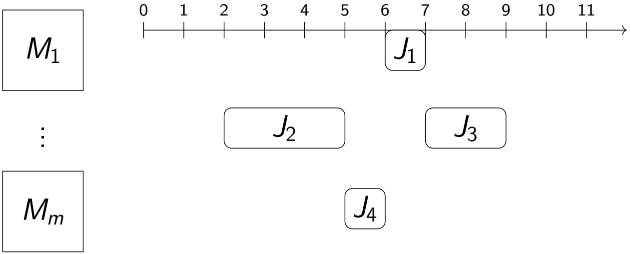
Outline

Scheduling



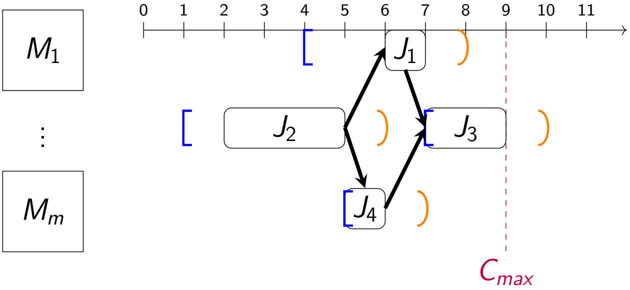
Set $\mathcal{J}$ of n jobs to schedule on m machines $M_1, \ldots, M_m$.

Scheduling



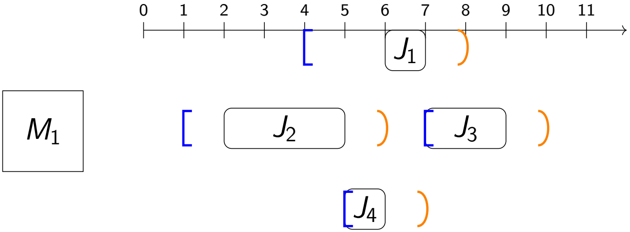
machines | job properties | optimization criterion

Scheduling



$$P|prec, r_j, d_j|C_{max}$$

Scheduling



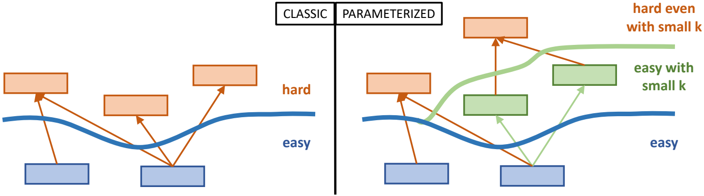
$1|r_j, d_j|C_{max}$ is strongly NP-hard [Lenstra et al. '77].

Motivation

- From the "My Ph.D. thesis in 180s" session at EDITE day (2023):

Complexité paramétrée et nouveaux schémas énumératifs efficaces pour le RCPSP

Maher MALLEM (LIP6, SU) sous la direction de Claire HANEN (UPL ; LIP6, SU)



Going beyond NP-hardness

- Problem $\mathcal{P}$, instance $\mathcal{I}$.
- P: deterministic time $\mathit{poly}(|\mathcal{I}|)$.
- NP: nondeterministic time $\mathit{poly}(|\mathcal{I}|)$.

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  - Problem $(\mathcal{P}, k)$, instance $\mathcal{I}$, parameter $k : \mathcal{P} \rightarrow \mathbb{N}_0$.
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  - FPT: deterministic time $f(k(\mathcal{I})) \cdot poly(|\mathcal{I}|)$.
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Fact [Flum, Grohe '06]

$(\mathcal{P}, k)$ is para-NP-hard if and only if $\mathcal{P}$ is NP-hard for some fixed value k_0 of parameter k.

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- Dynamic Programming

High-multiplicity setting

Definition

Input of $P|HM, r_j, d_j|X$:

- integers m and d,
- integer vector $(n_i, p_i, r_i, d_i)_{1 \leq i \leq d}$.

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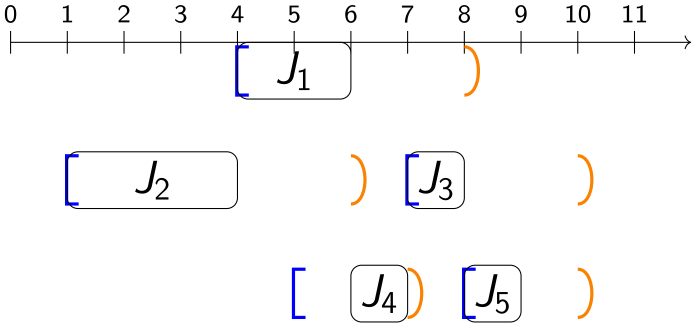
Theorem [Knop et al. '19]

Let $\mathcal{X}_{lin} = \{C_{max}, C_{min}, F_{max}, L_{max}, \sum w_j U_j\}$.
Let $\mathcal{X}_{poly} = \{\sum w_j C_j, \sum w_j F_j, \sum w_j T_j, \ell_p - norm\}$.

$P|HM, r_j, d_j|X$ is FPT parameterized by:

- $d + p_{max}$ if $X \in \mathcal{X}_{lin} \cup \mathcal{X}_{poly}$,
- $d + m$ if $X \in \mathcal{X}_{lin}$.

Outline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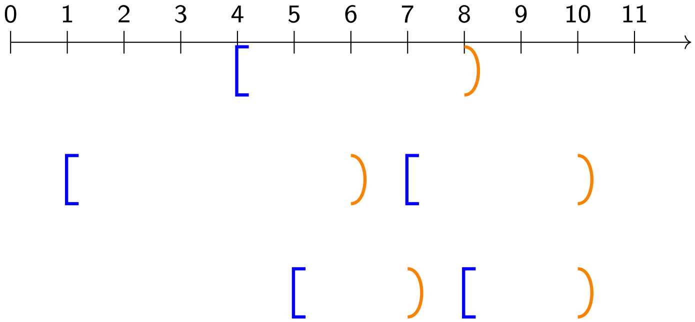
Pathwidth μ



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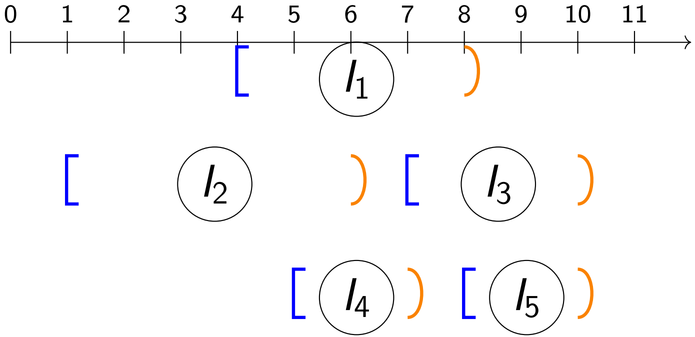

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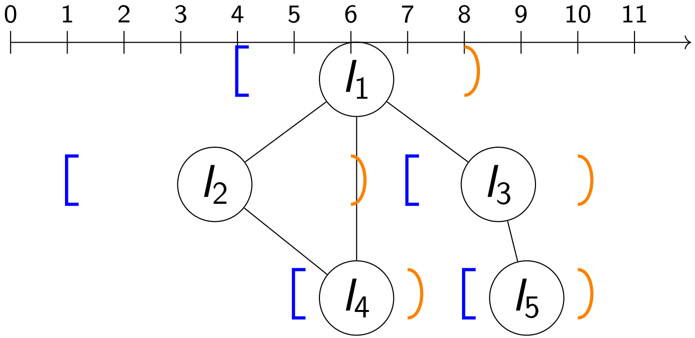

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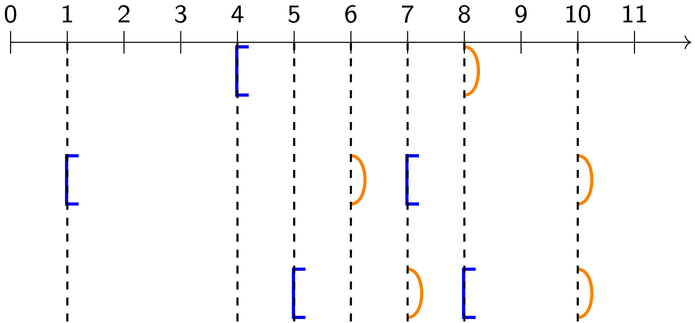

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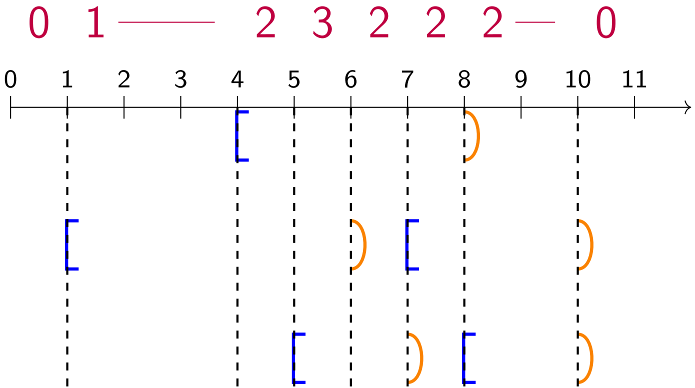

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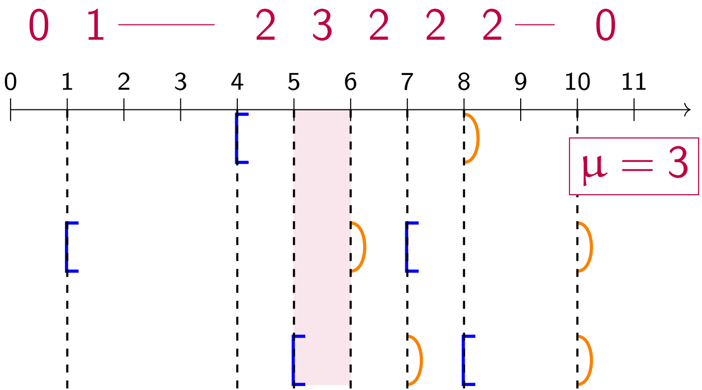

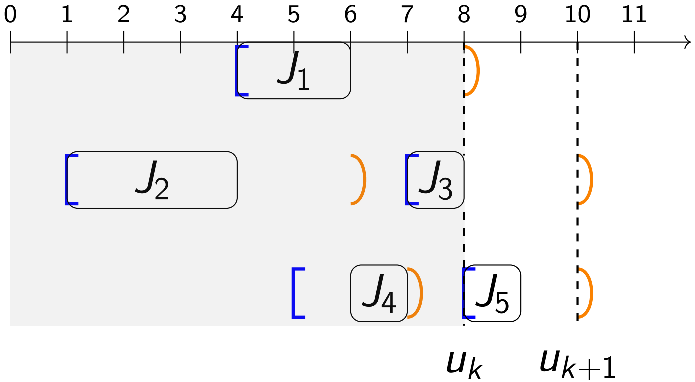
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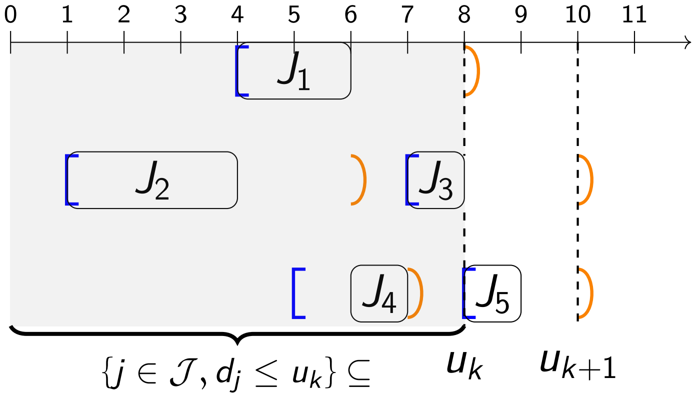
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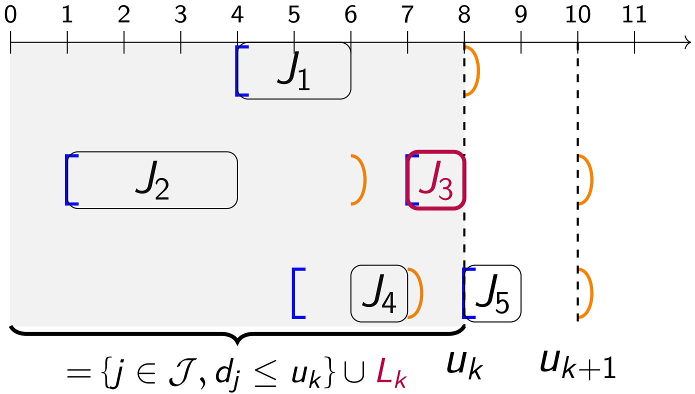
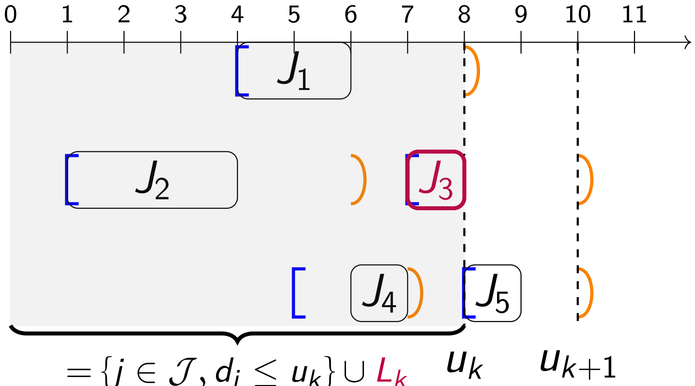


- $J_j \in L_k \implies [u_k, u_{k+1}) \subseteq [r_j, d_j)$: at most μ such jobs.

Pathwidth μ



$$\underbrace{= \underbrace{\{j \in \mathcal{J}, d_j \leq u_k\}}_{= Z_k} \cup L_k}$$

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Dynamic programming with pathwidth μ

Dynamic programming state (k, L_k)

- u_k = associated release date/deadline value.
- $Z_k \cup L_k$ = set of jobs starting before u_k.
- State value = minimum makespan among the associated partial schedules.

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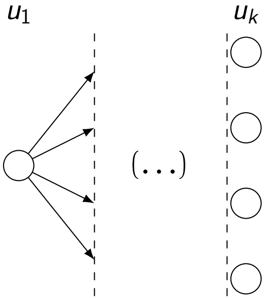

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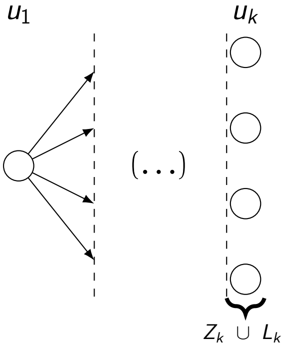

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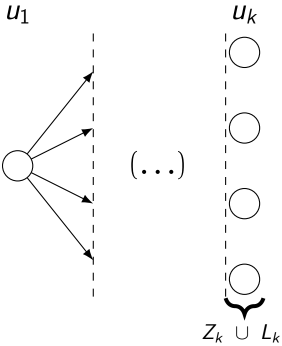

$\leq 2^{\mu}$ states

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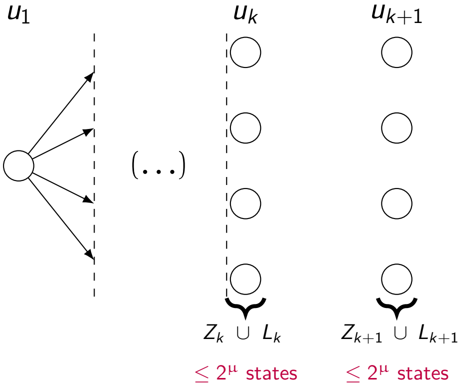

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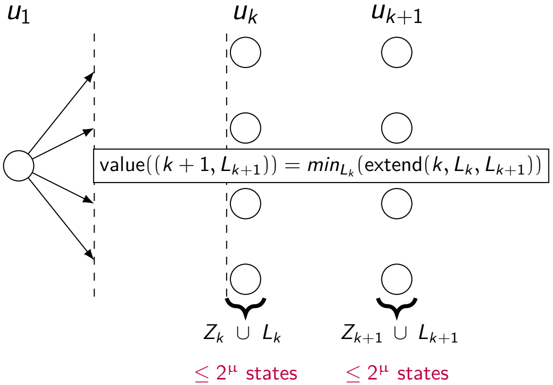

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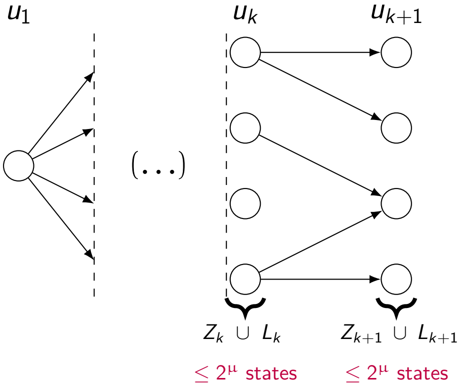

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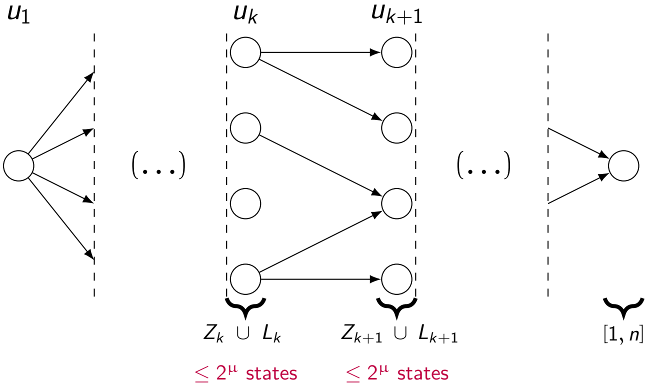


Algorithm 1: main()

▷ Solving $1|prec, r_j, d_j|C_{max}$.

preprocessing() ▷ Compute sequence $(u_k)_{1\leq k\leq K}$. Compute associated job subsets.
$u_{K+1} \leftarrow \infty$
Create table T with K columns (1 to K) and 2^{μ} slots in each column k.
Initialize T with ∞ everywhere.
$T[1, \emptyset] \leftarrow 0$
for $k = 1$ to $K - 1$ **do**
  for each (L_k, L_{k+1}) **do**
    $T[k+1, L_{k+1}] \leftarrow \min(T[k+1, L_{k+1}], \text{extend}(k, L_k, L_{k+1}))$
return $T[K, \emptyset]$

Algorithm 2: extend(k, L_k, L_{k+1})

$Cmax \leftarrow \max(u_k, T[k, L_k])$
if $Cmax = \infty$ **then return** ∞
▷ Ensure that all jobs in L_k with a deadline higher than u_{k+1} are still in L_{k+1}.
if $(L_k - Z_{k+1}) \not\subseteq L_{k+1}$ **then return** ∞
▷ Add jobs to the existing partial schedule.
$mandatory \leftarrow Z_{k+1} \cap \{J_j \in \mathcal{J}, [u_k, u_{k+1}) \subseteq [r_j, d_j)\}$
$add \leftarrow (mandatory \cup L_{k+1}) - L_k$
sort$(add, (prec, \text{nondecreasing } d_j))$
for j in add (in order) **do**
    if appending J_j would violate a precedence constraint **then return** ∞
    $Cmax \leftarrow Cmax + p_j$
    if $Cmax > d_j$ **then return** ∞
▷ Ensure that all the added jobs start earlier than u_{k+1}.
$last \leftarrow$ last job in add.
if $Cmax - p_{last} \geq u_{k+1}$ **then return** ∞
return $Cmax$

Main result

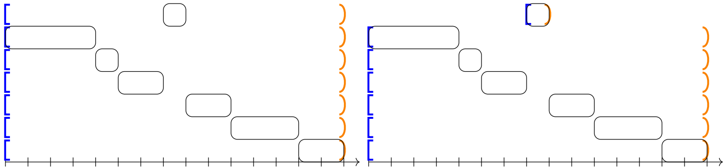
Theorem [Hanen, Mallem, Munier '22]

$1|prec, r_j, d_j|C_{max}$ can be solved in time:

$$\mathcal{O}(\mu^2 \cdot 4^{\mu} \cdot n + n^2).$$

Improving on Pathwidth

- Two single machine instances with time windows.



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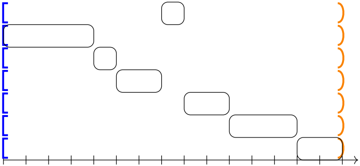
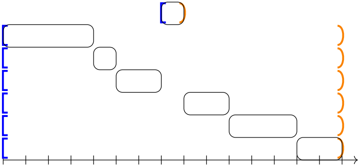


$\mu = n$



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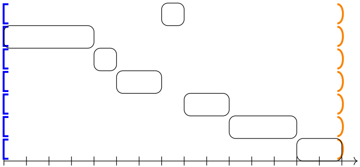

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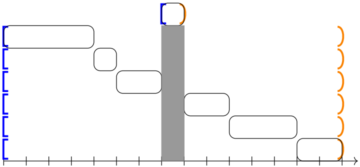

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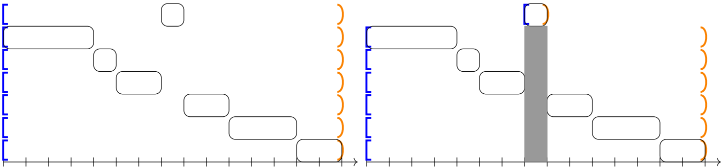

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Definition: Proper sets and proper level

- $\forall J_i \in \mathcal{J}, \Pi_i = \{J_j \in \mathcal{J}, r_j < r_i \wedge d_i < d_j\}$
- $q = max_{J_i \in \mathcal{J}} \; |\Pi_i|$

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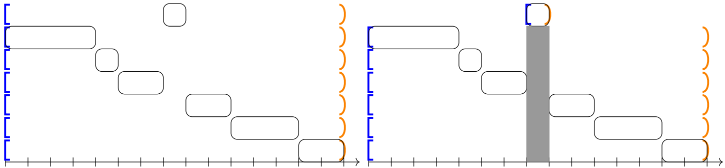

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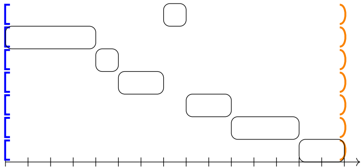
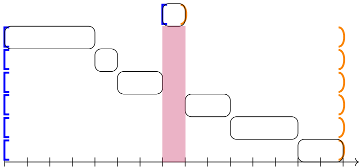


$\mu = n$

$q = 0$



$\mu = n$

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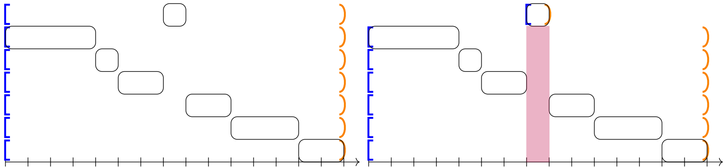

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Theorem [Mallem, Hanen, Munier '24]

$1|r_j, d_j|C_{max}$ can be solved in time:

$$\mathcal{O}((q+1)^2 \cdot 4^q \cdot n + n^2).$$

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Theorem [Hanen, Munier '23]

$P2|r_j, d_j|C_{max}$ is para-NP-hard parameterized by μ (with $\mu = 4$).

Pathwidth is not enough

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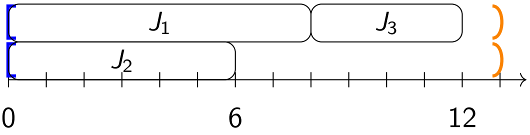
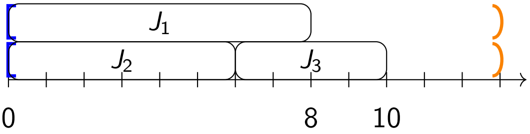
  - Example: $(6, 12)$ vs. $(8, 10)$.





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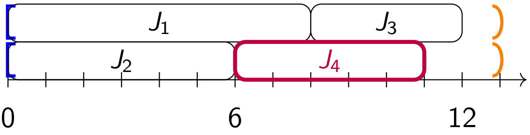
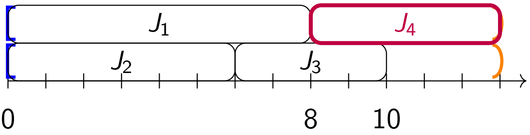


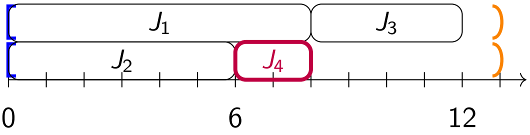


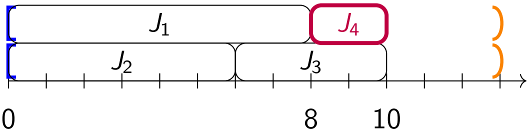
Pathwidth is not enough

Theorem [Hanen, Munier '23]

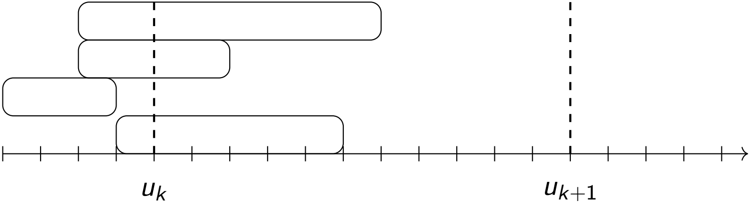
$P2|r_j, d_j|C_{max}$ is para-NP-hard parameterized by μ (with $\mu = 4$).

- For each job subset: Pareto front of makespan tuples.
  - Example: $(6, 12)$ vs. $(8, 10)$.





Bounding the number of border schedules



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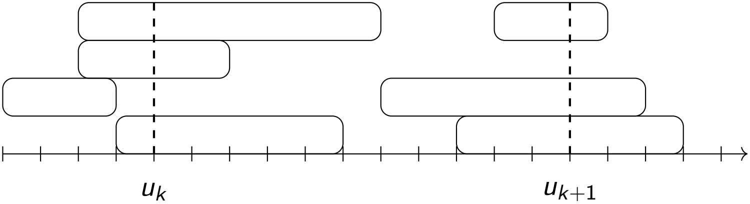

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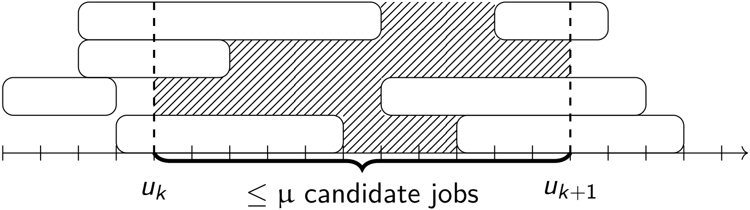

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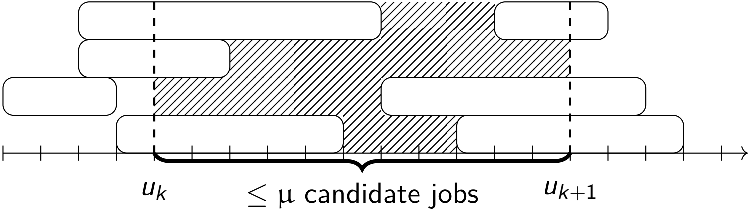

Parameter definitions

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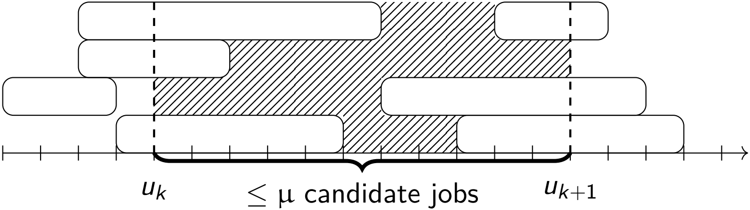

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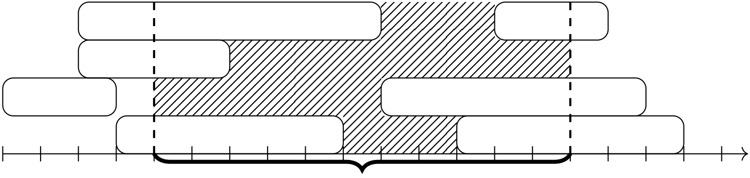

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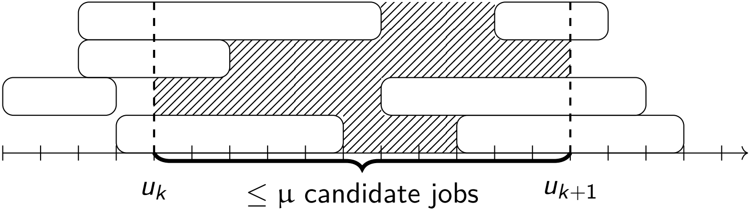

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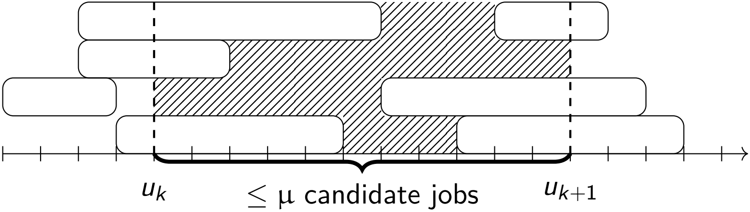

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- $\mathcal{O}((\mu\beta + 1)^m)$ "border schedules" B_k associated to u_k.
- $\mathcal{O}(2^\mu \cdot (\mu\beta + 1)^\mu)$ states (k, L_k, B_k) associated to u_k.

Results on parallel machines

Theorem [Tarhan et al. '23]

$P|prec, r_j, d_j|C_{max}$ is FPT parameterized by $\mu + p_{max}$.

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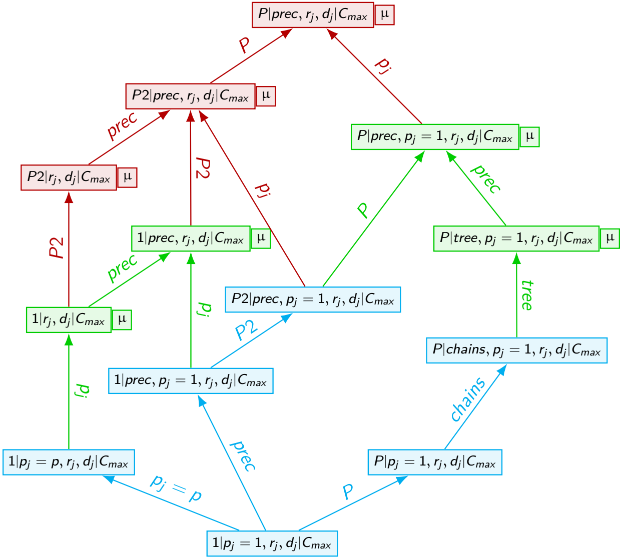
Theorem [Mallem, Hanen, Munier '25 (submitted)]

$P|prec, p_j = 1, r_j, d_j|C_{max}$ is para-NP-hard parameterized by $q + C_{max}$ (with $q = 0$ and $C_{max} = 3$).

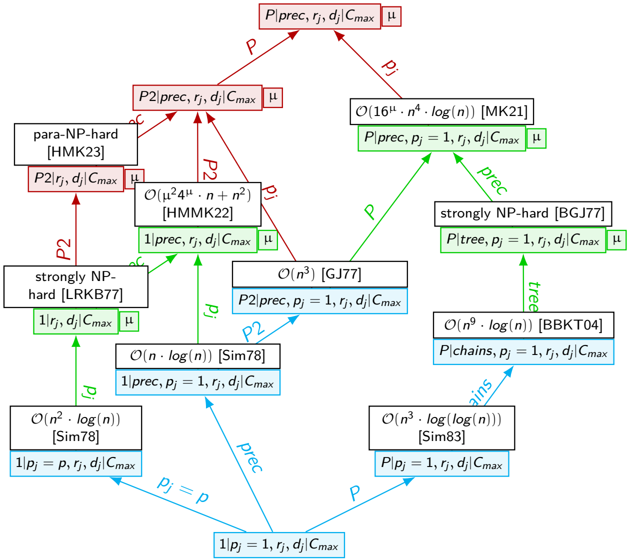
Outline

1. Introduction
   - The Problem
   - Parameterized Complexity
2. With a single machine
   - Parameter Pathwidth
   - Dynamic Programming
   - Improving the Pathwidth: Parameter Proper Level
3. With Parallel Identical Machines
   - Pathwidth is not Enough
   - Border Enumeration
4. Conclusion

Conclusion


$P|prec, r_j, d_j|C_{max}$ µ
P
p_j
$P2|prec, r_j, d_j|C_{max}$ µ
$prec$
$P2$
p_j
$P|prec, p_j = 1, r_j, d_j|C_{max}$ µ
$P2|r_j, d_j|C_{max}$ µ
P
$prec$
$P2$
$1|prec, r_j, d_j|C_{max}$ µ
$P|tree, p_j = 1, r_j, d_j|C_{max}$ µ
$prec$
$1|r_j, d_j|C_{max}$ µ
p_j
$P2|prec, p_j = 1, r_j, d_j|C_{max}$
$tree$
$P2$
$P|chains, p_j = 1, r_j, d_j|C_{max}$
$1|prec, p_j = 1, r_j, d_j|C_{max}$
p_j
$chains$
$prec$
$1|p_j = p, r_j, d_j|C_{max}$
$P|p_j = 1, r_j, d_j|C_{max}$
$p_j = p$
P
$1|p_j = 1, r_j, d_j|C_{max}$

Conclusion


$P|prec, r_j, d_j|C_{max}$ μ
P
pj
$P2|prec, r_j, d_j|C_{max}$ μ
$\mathcal{O}(16^{\mu} \cdot n^4 \cdot log(n))$ [MK21]
$P|prec, p_j = 1, r_j, d_j|C_{max}$ μ
para-NP-hard [HMK23]
$P2|r_j, d_j|C_{max}$ μ
P2
$\mathcal{O}(\mu^2 4^{\mu} \cdot n + n^2)$ [HMMK22]
$1|prec, r_j, d_j|C_{max}$ μ
prec
strongly NP-hard [BGJ77]
$P|tree, p_j = 1, r_j, d_j|C_{max}$ μ
strongly NP-hard [LRKB77]
$1|r_j, d_j|C_{max}$ μ
$\mathcal{O}(n^3)$ [GJ77]
$P2|prec, p_j = 1, r_j, d_j|C_{max}$
tree
$\mathcal{O}(n^9 \cdot log(n))$ [BBKT04]
$P|chains, p_j = 1, r_j, d_j|C_{max}$
$\mathcal{O}(n \cdot log(n))$ [Sim78]
$1|prec, p_j = 1, r_j, d_j|C_{max}$
$\mathcal{O}(n^2 \cdot log(n))$ [Sim78]
$1|p_j = p, r_j, d_j|C_{max}$
$\mathcal{O}(n^3 \cdot log(log(n)))$ [Sim83]
$P|p_j = 1, r_j, d_j|C_{max}$
$p_j = p$
P
$1|p_j = 1, r_j, d_j|C_{max}$

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  [Baart et al. '21] $1|r_j, \bar{d}_j, s_{j,k}, reject|\sum_{J_j accepted}(v_j - w_j T_j)$

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  Example:
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  - FPT exact resolution with respect to $\mu + \sigma$: $\mathcal{O}(\sigma\mu^2\, 2^{\mu} \cdot n^2)$.
  - FPT *FPTAS* with respect to μ: $\mathcal{O}(\mu^2\, 2^{pw} \cdot \frac{n^3}{\varepsilon^2})$.

Parameterized complexity classes

nondet. $f(k) \cdot poly(|\mathcal{I}|)$ time

det. $|\mathcal{I}|^{f(k)}$ time

para-NP

XP

$XNLP$

nondet. $f(k) \cdot poly(|\mathcal{I}|)$ time
& $f(k) \cdot log(n)$ space

$\cdots$

$W[2]$

$W[1]$

FPT

det. $f(k) \cdot poly(|\mathcal{I}|)$ time

Parameterized complexity classes

nondet. $f(k) \cdot poly(|\mathcal{I}|)$ time

det. $|\mathcal{I}|^{f(k)}$ time

$\boxed{k\text{-COLORING}}$ para-NP

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$\dots$

$\boxed{k\text{-DOMINATING SET}}$ $W[2]$

$\boxed{k\text{-CLIQUE}}$ $W[1]$

FPT

det. $f(k) \cdot poly(|\mathcal{I}|)$ time

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