Kernelization Algorithms on Machine Scheduling with Time Windows

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Outline

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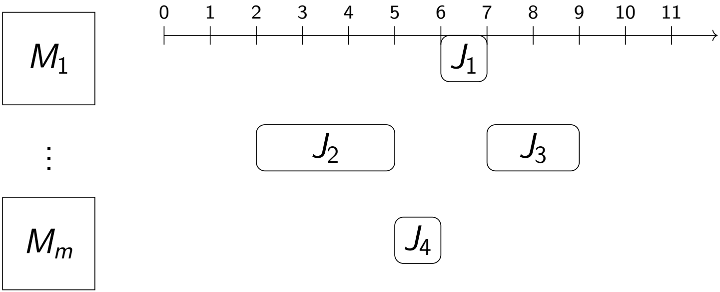
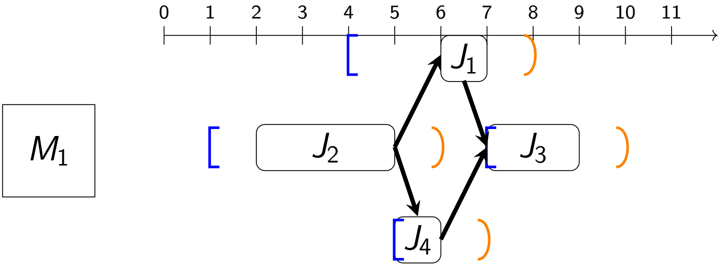
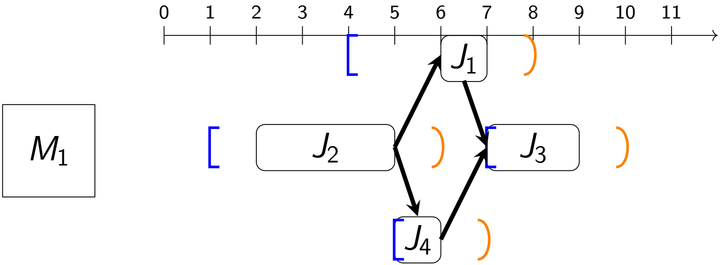
1. Introduction
   - The Problem
   - Parameterized Complexity
2. Polynomial Kernel with Vertex Cover
   - Overview
   - Deleting Jobs
   - Reducing Time Values
3. Kernels with Pathwidth
   - No Polynomial Kernel
   - Exponential Size Kernels
4. Conclusion

Machine Scheduling





$$1|prec, r_j, d_j|\star$$



$1|prec, r_j, d_j|\star$ is strongly NP-hard [LRKB 77].

Going beyond NP-hardness

- Problem $\mathcal{P}$, instance $\mathcal{I}$.

- P: deterministic time $poly(|\mathcal{I}|)$.

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- Kernelization algorithm: outputs (I', k') in time $poly(|\mathcal{I}|)$ such that:
    - I' is equivalent to I.
    - $|I'| + k' \leq f(k)$ for some computable function f.

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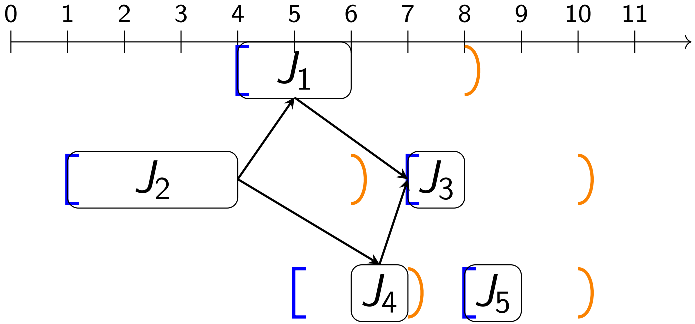
- Polynomial kernel: f is polynomial.

Pathwidth *pw* and vertex cover *vc*



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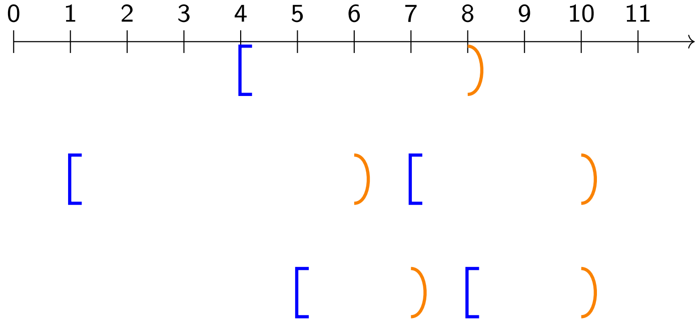
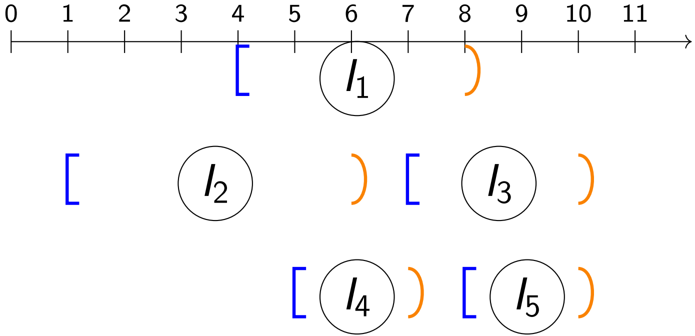


Pathwidth pw and vertex cover vc



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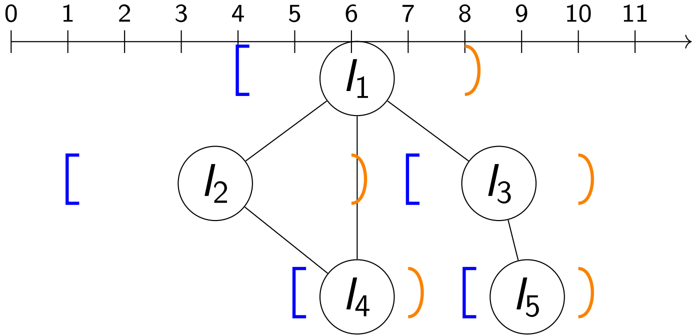

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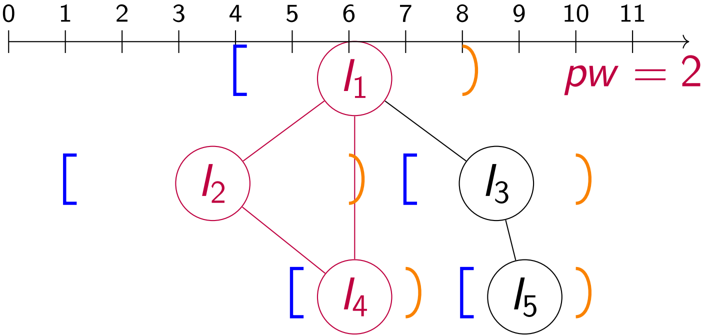

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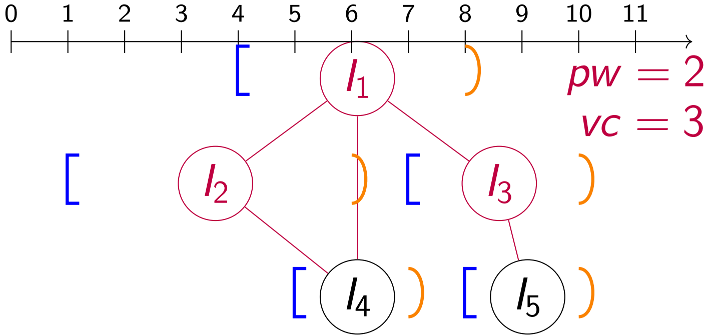

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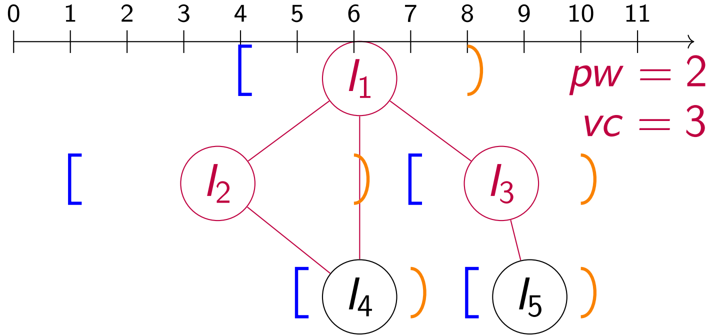

Result

- $pw \leq vc$
- [Mallem, Hanen, Munier-Kordon 22]
  $1|prec, r_j, d_j|\star$ in time $\mathcal{O}((pw+1) \cdot 4^{pw} \cdot n + n \cdot log(n))$.

Outline

Polynomial kernel overview

Proposition

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Instance I		Instance I'		Instance I''
n jobs	$\xrightarrow{(1)}$	$\mathcal{O}(vc^2)$ jobs	$\xrightarrow{(2)}$	$\mathcal{O}(vc^2)$ jobs
size $\mathcal{O}(n \cdot log(max_j(d_j)))$		size $\mathcal{O}(vc^2 \cdot log(max_j(d_j)))$		size $\mathcal{O}(vc^8)$

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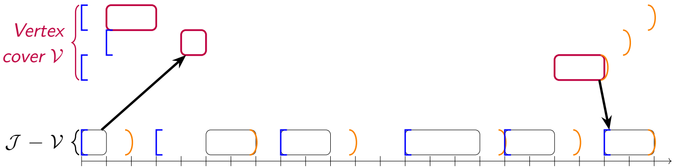
- (1) Delete *most* jobs.
  - Use a reduction rule.
  - Time $\mathcal{O}(n^2)$.
- (2) Reduce time values.
  - Translate into a BIP and use a theorem from [Frank, Tardos 87].
  - Time $\mathcal{O}(n^{18})$.

(1) Viewing non-vertex covering jobs as gaps



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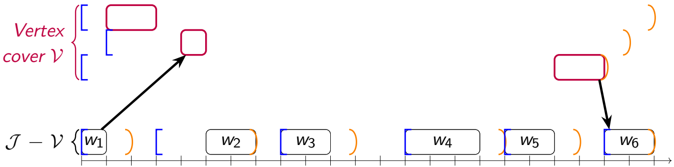

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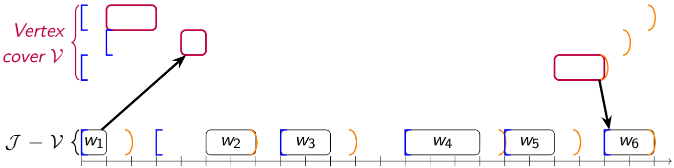

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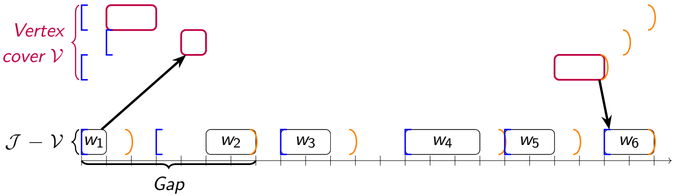

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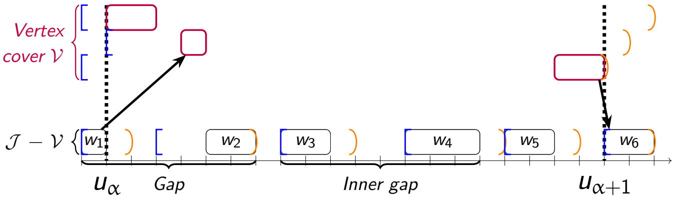

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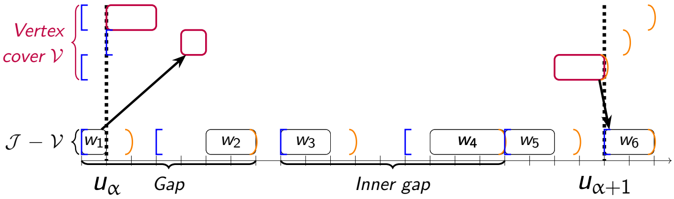

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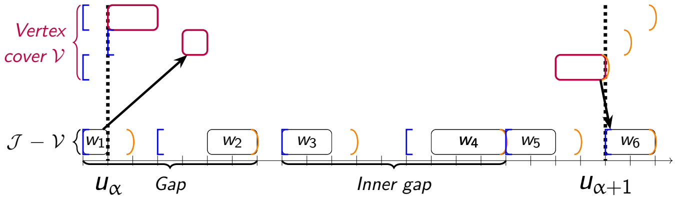

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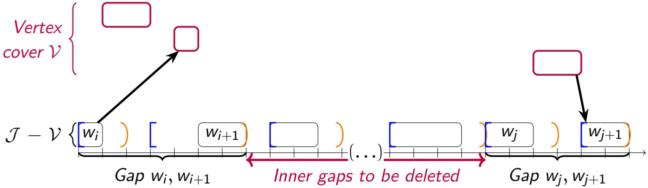
- Maximum capacity: w_i at the earliest, w_{i+1} at the latest.
- $vc_\alpha = |\{J_j \in \mathcal{V}, [u_\alpha, u_{\alpha+1}) \subseteq [r_j, d_j)\}|$.

Reduction rule

In each interval $[u_\alpha, u_{\alpha+1})$, only keep the $3vc_\alpha$ inner gaps with highest maximum capacity.

(1) Deleting unnecessary inner gaps



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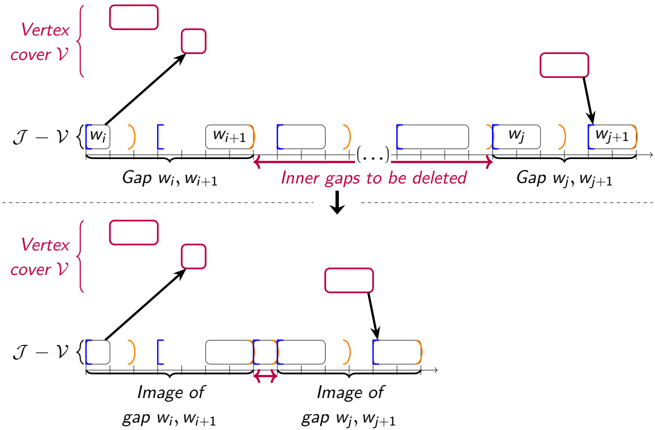

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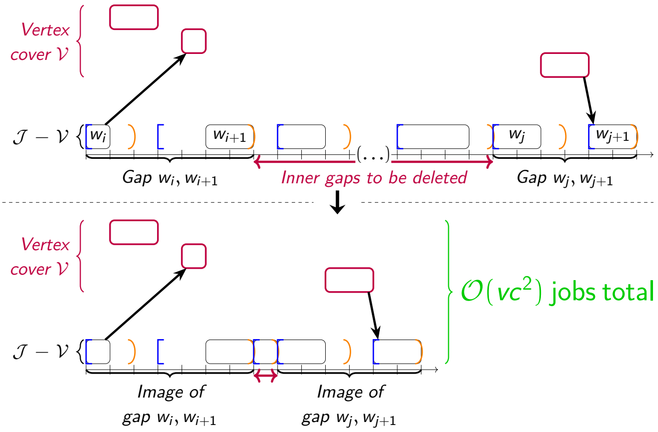

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Theorem [Frank, Tardos 87]

Let $\mathtt{w} = (w_1, \ldots, w_d)$ be a vector of rational numbers and let N be a positive integer. Then in time $\mathcal{O}(d^8 \cdot (d + log(N)))$ one can build a vector $\mathtt{w}' = (w'_1, \ldots, w'_d)$ of positive integers such that:

- $w'_i \leq 2^{4d^3} \cdot N^{d(d+2)}$ for all $1 \leq i \leq d$,
- for every $\mathtt{x} = (x_1, \ldots, x_d)$ vector of integers such that $\sum_{i=1}^{d} |x_i| \leq N - 1$:

$$\sum_{i=1}^{d} w_i x_i \geq 0 \text{ if and only if } \sum_{i=1}^{d} w'_i x_i \geq 0.$$

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  - At most $n + 2$ variables equal to one in each equation of any solution.

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- Use the theorem on instance I' with $\tilde{n} = \mathcal{O}(vc^2)$ jobs.
  - $d = 3\tilde{n}$, $\mathtt{w} = (p_1, r_1, d_1, \ldots, p_{\tilde{n}}, r_{\tilde{n}}, d_{\tilde{n}})$, $N = \tilde{n} + 3$.

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  - Time $\mathcal{O}(vc^{18}) = \mathcal{O}(n^{18})$.
  - Each reduced time value taking space $\mathcal{O}(vc^6)$.

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  - $pw \leq poly(max_i(|P_i|) + log(t))$

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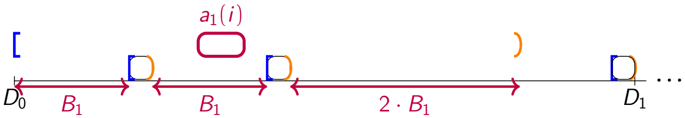

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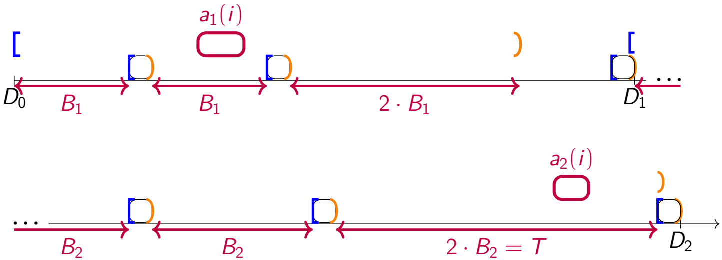

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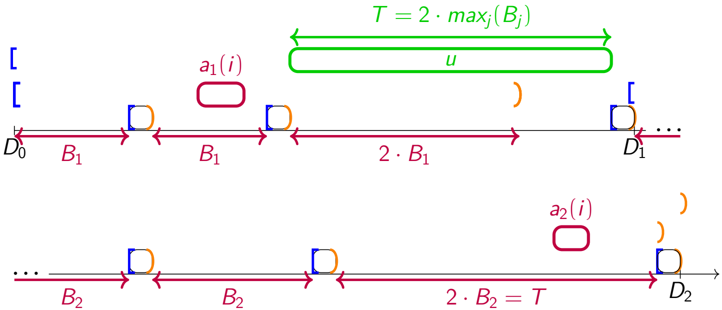

Algorithm 1 NaiveKernel($1|prec, r_j, d_j|\star$) with pathwidth pw

Input: an instance I of $1|prec, r_j, \bar{d}_j|\star$.
Compute pw in time $\mathcal{O}(n \cdot log(n))$.
if $|I| \leq 2^{pw}$ **then**
  return I
else
  Use the *FPT* algorithm from [MHMK 22] in time $\mathcal{O}((pw+1)^2 \cdot 4^{pw} \cdot n + n \cdot log(n)) = \mathcal{O}(|I|^2 \cdot log(|I|) \cdot n)$.
  if the output is YES **then**
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An exponential size kernel w.r.t. pathwidth pw (cont.)

Algorithm 2 Kernel($1|prec, r_j, d_j|\star$) with pathwidth pw

```
Input: an instance I of 1|prec, r_j, \bar{d}_j|⋆.
Compute pw and vc in time O(n · log(n)).
if vc ≤ 2^pw then
    return the polynomial kernel of I w.r.t. vc.
else
    Use the FPT algorithm from [MHMK 22] in time
    O((pw + 1)^2 · 4^pw · n + n · log(n)) = O(vc^2 · log(vc) · n).
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Parameter	$1\|prec, r_j, d_j\|\star$	$P2\|r_j, d_j\|\star$
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Future work

- Investigate other scheduling problems with time windows.

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- Investigate other scheduling problems with time windows.
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- Improve the existing kernels.

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- [Lasserre, Queyranne 92] Binary IP equivalent to $1|prec, r_j, d_j|\star$.

$$BIP(I): \begin{cases} \text{One job per position} & \sum_{j=1}^{n} x_{i,j} = 1 \text{ for all } 1 \leq i \leq n. \\ \text{One position per job} & \sum_{i=1}^{n} x_{i,j} = 1 \text{ for all } 1 \leq j \leq n. \\ \text{Precedence constraints} & \sum_{i=1}^{i_0} (x_{i,j_1} - x_{i,j_2}) \geq 0 \text{ for all } 1 \leq i_0 \leq n,\ (j_1, j_2) \in A. \\ \text{Time window checks} & \sum_{j=1}^{n} r_j \cdot x_{i_1,j} + \sum_{i=i_1}^{i_2} \sum_{j=1}^{n} p_j \cdot x_{i,j} \leq \sum_{j=1}^{n} \bar{d}_j \cdot x_{i_2,j} \\ & \text{for all } 1 \leq i_1 \leq i_2 \leq n. \end{cases}$$

(2) Reducing time values (with the BIP)

Theorem [Frank, Tardos 87]

Let $\mathtt{w} = (w_1, \ldots, w_d)$ be a vector of rational numbers and let N be a positive integer. Then in time $\mathcal{O}(d^8 \cdot (d + log(N)))$ one can build a vector $\mathtt{w}' = (w'_1, \ldots, w'_d)$ of positive integers such that:

- $w'_i \leq 2^{4d^3} \cdot N^{d(d+2)}$ for all $1 \leq i \leq d$,
- for every $\mathtt{x} = (x_1, \ldots, x_d)$ vector of integers such that $\sum_{i=1}^{d} |x_i| \leq N - 1$:

$$\sum_{i=1}^{d} w_i x_i \geq 0 \text{ if and only if } \sum_{i=1}^{d} w'_i x_i \geq 0.$$

- [Lasserre, Queyranne 92] Binary IP equivalent to $1|prec, r_j, d_j|\star$.

$$BIP(I) : \begin{cases} \text{One job per position} & \sum_{j=1}^{n} x_{i,j} = 1 \text{ for all } 1 \leq i \leq n. \\ \text{One position per job} & \sum_{i=1}^{n} x_{i,j} = 1 \text{ for all } 1 \leq j \leq n. \\ \text{Precedence constraints} & \sum_{i=1}^{i_0} (x_{i,j_1} - x_{i,j_2}) \geq 0 \text{ for all } 1 \leq i_0 \leq n,\ (j_1, j_2) \in A. \\ \text{Time window checks} & \sum_{j=1}^{n} r_j \cdot x_{i_1,j} + \sum_{i=i_1}^{i_2} \sum_{j=1}^{n} p_j \cdot x_{i,j} \leq \sum_{j=1}^{n} \bar{d}_j \cdot x_{i_2,j} \\ & \text{for all } 1 \leq i_1 \leq i_2 \leq n. \end{cases}$$

- Use the theorem on instance I' with $\tilde{n} = \mathcal{O}(vc^2)$ jobs.
  - $d = 3\tilde{n}$, $\mathtt{w} = (p_1, r_1, d_1, \ldots, p_{\tilde{n}}, r_{\tilde{n}}, d_{\tilde{n}})$, $N = \tilde{n} + 3$.