Scheduling meets parameterized complexity

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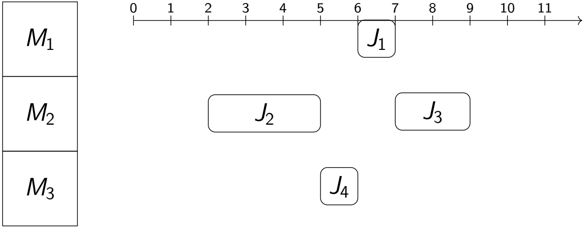
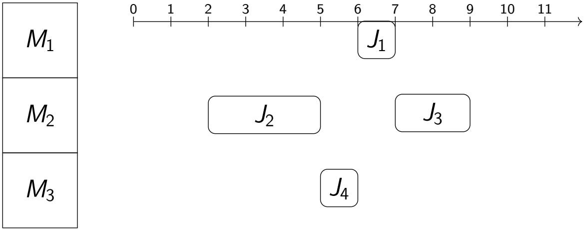
Parallel machine scheduling

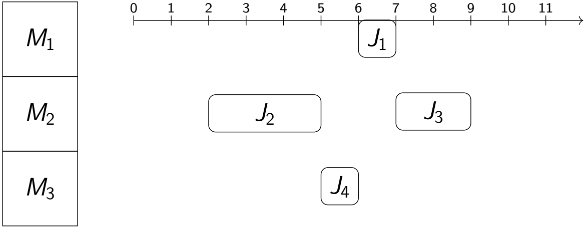


Parallel machine scheduling



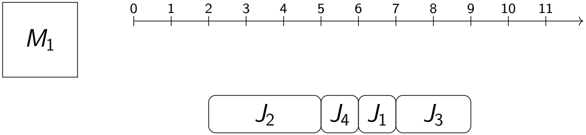
machines | job properties | optimization criterion

Parallel machine scheduling



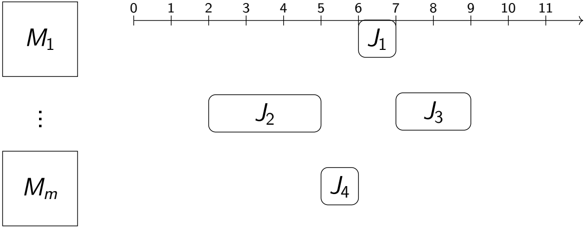
$$P3|.|.$$

Parallel machine scheduling



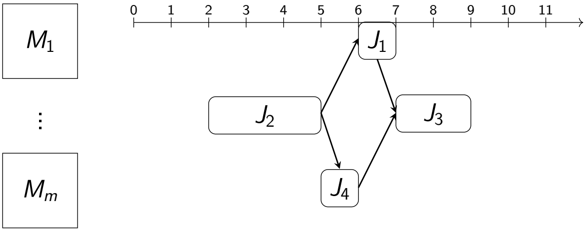
$$1|.|.$$

Parallel machine scheduling

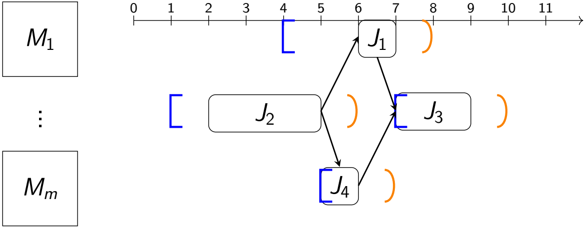


$P|.|.$

Parallel machine scheduling

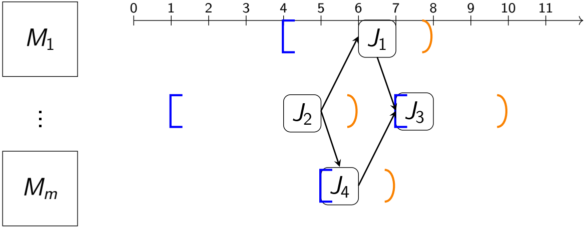


$$P|prec|.$$



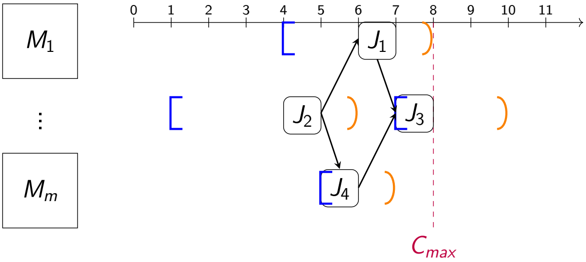
$$P|prec, r_j, d_j|.$$

Parallel machine scheduling



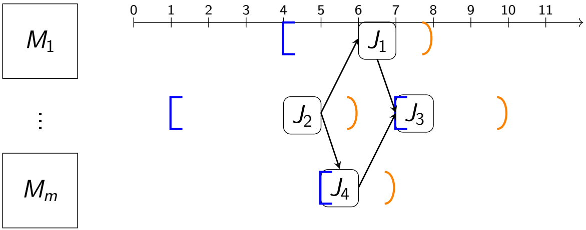
$$P|prec, p_j = 1, r_j, d_j|.$$

Parallel machine scheduling



$$P|prec, p_j = 1, r_j, d_j|C_{max}$$

Parallel machine scheduling



$$P|prec, p_j = 1, r_j, d_j|\star$$

Exploring complexity frontiers

From poly-time solvable problem $1|p_j = 1, r_j, d_j|\star$

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- Add precedence relations *prec*

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- Add precedence delays $\ell_{i,j}$

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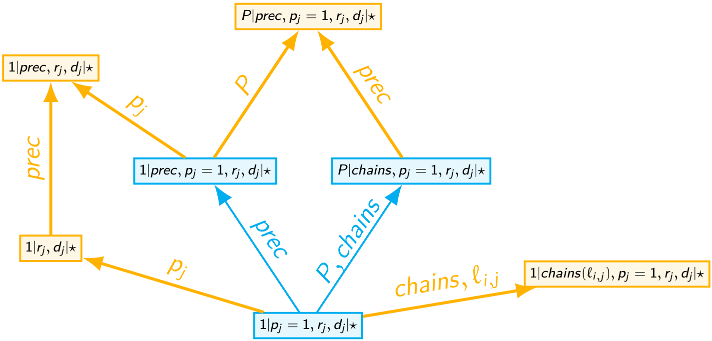
- $p_j = 1 \longrightarrow$ any p_j
- Add precedence relations *prec*
- Add precedence delays $\ell_{i,j}$

$1|r_j, d_j|\star$ is strongly NP-hard. [Brucker, Kan, Lenstra 1977]

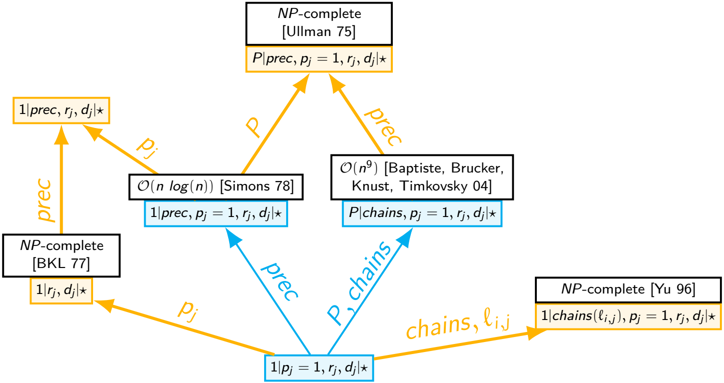
Classical complexity theory

$\boxed{1|p_j = 1, r_j, d_j|\star}$

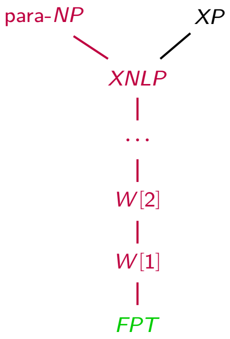
Classical complexity theory



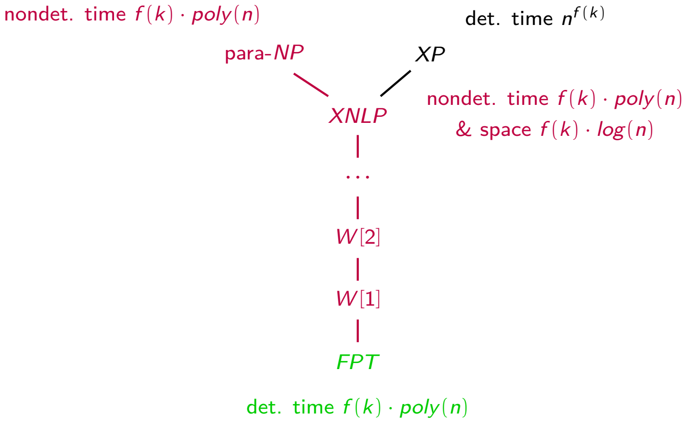
Classical complexity theory



Parameterized complexity classes



Parameterized complexity classes



Parameterized complexity in machine scheduling

- [Bodlaender, Fellows 1995]
  - #machines m: $P|prec, p_j = 1|C_{max}$ is W[2]-hard.

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- [Mnich, Wiese 2015]
  - Maximum processing time p_{max}: $P||C_{max} \in$ FPT.

Parameterized complexity in machine scheduling

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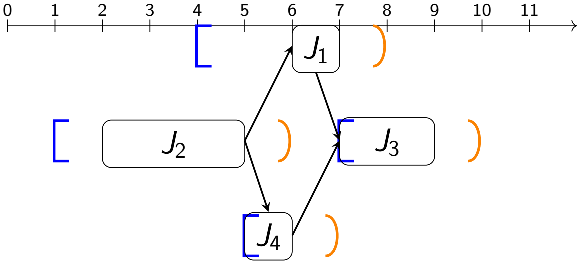
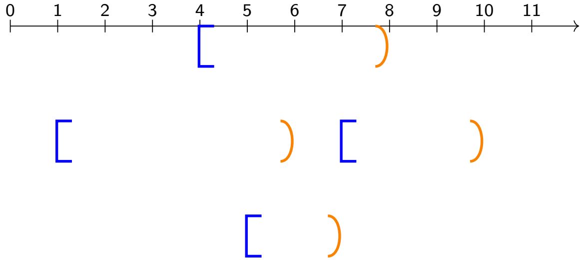
  - #machines m: $P|prec, p_j = 1|C_{max}$ is W[2]-hard.
- [Mnich, Wiese 2015]
  - Maximum processing time p_{max}: $P||C_{max} \in$ FPT.
- [Hermelin, Karhi, Pinedo, Shabtay 2018]
  - #distinct d_j, #distinct p_j, #distinct w_j: $1||\sum w_j U_j \in$ FPT with two of the three.

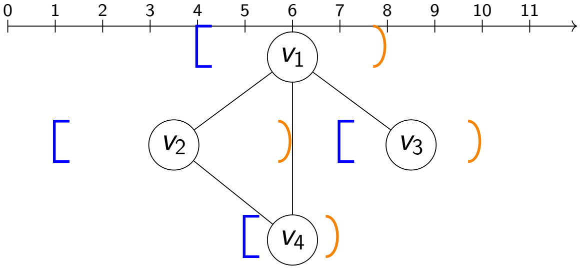
Pathwidth μ



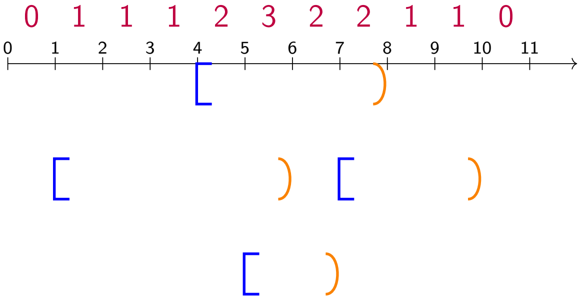
Pathwidth μ



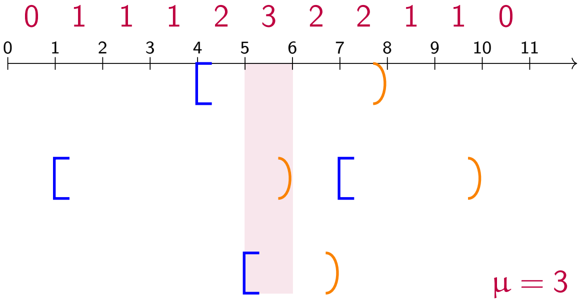
Pathwidth μ



Pathwidth μ

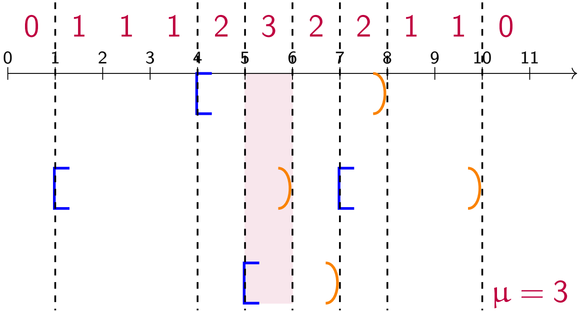


Pathwidth μ

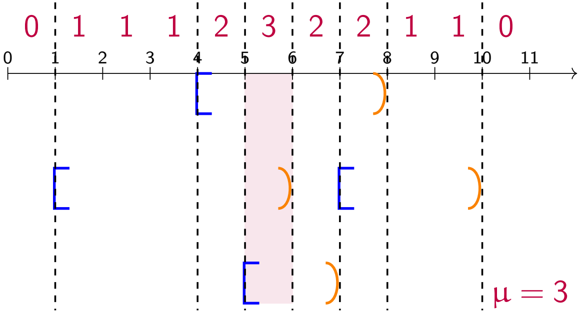


$\mu = 3$

Pathwidth μ

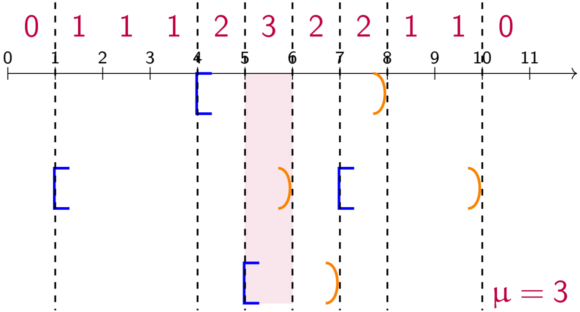


Pathwidth μ



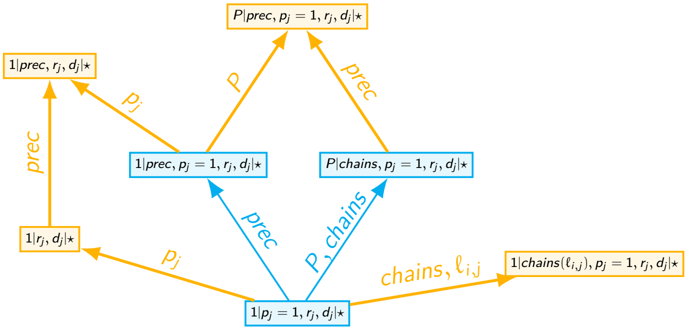
- [Munier 2021] $P|prec, p_j = 1, r_j, d_j|C_{max}$ is FPT parameterized by μ.

Pathwidth μ

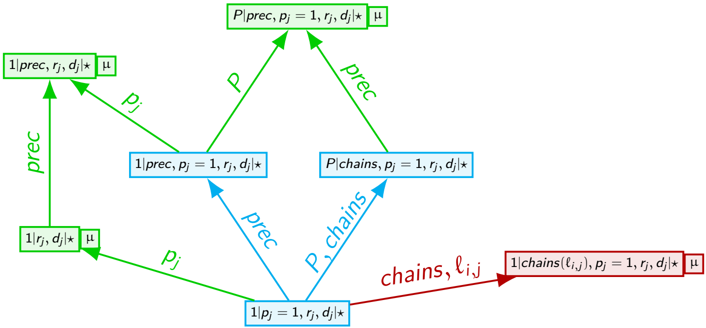


- [Munier 2021] $P|prec, p_j = 1, r_j, d_j|C_{max}$ is FPT parameterized by μ.
- [Bodlaender, Groenland, Jacob, Jaffke, Lima 2022] $P|prec, p_j = 1|C_{max}$ is XNLP-complete parameterized by $m + w$.

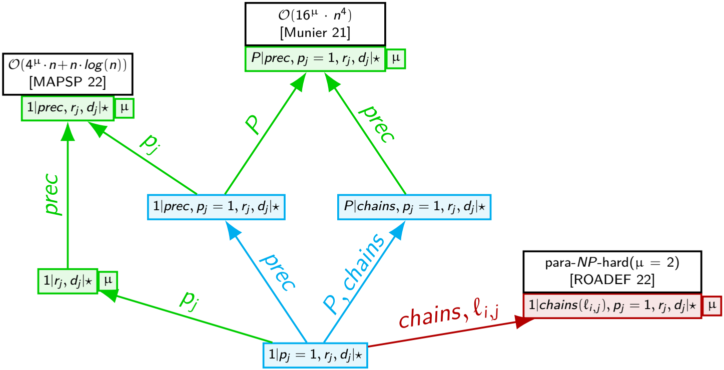
Parameter μ


$P|prec, p_j = 1, r_j, d_j|\star$
$1|prec, r_j, d_j|\star$
$1|prec, p_j = 1, r_j, d_j|\star$
$P|chains, p_j = 1, r_j, d_j|\star$
$1|r_j, d_j|\star$
$1|chains(\ell_{i,j}), p_j = 1, r_j, d_j|\star$
$1|p_j = 1, r_j, d_j|\star$
P
prec
p_j
prec
prec
P, chains
p_j
chains, $\ell_{i,j}$

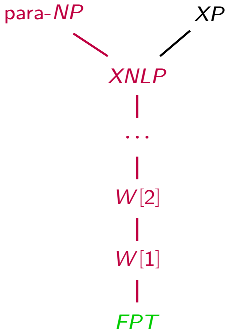
Parameter μ


$P|prec, p_j = 1, r_j, d_j|\star$ μ
$1|prec, r_j, d_j|\star$ μ
p_j
P
$prec$
$prec$
$1|prec, p_j = 1, r_j, d_j|\star$
$P|chains, p_j = 1, r_j, d_j|\star$
$1|r_j, d_j|\star$ μ
$prec$
$P, chains$
p_j
$chains, \ell_{i,j}$
$1|chains(\ell_{i,j}), p_j = 1, r_j, d_j|\star$ μ
$1|p_j = 1, r_j, d_j|\star$

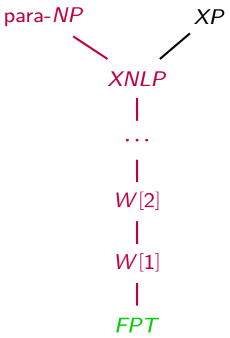
Parameter μ


$\mathcal{O}(16^{\mu} \cdot n^4)$
[Munier 21]
$P|prec, p_j = 1, r_j, d_j|\star$
μ
$\mathcal{O}(4^{\mu} \cdot n + n \cdot log(n))$
[MAPSP 22]
$1|prec, r_j, d_j|\star$
μ
prec
p_j
P
prec
$1|prec, p_j = 1, r_j, d_j|\star$
$P|chains, p_j = 1, r_j, d_j|\star$
$1|r_j, d_j|\star$
μ
p_j
prec
P, chains
para-NP-hard(μ = 2)
[ROADEF 22]
chains, $\ell_{i,j}$
$1|chains(\ell_{i,j}), p_j = 1, r_j, d_j|\star$
μ
$1|p_j = 1, r_j, d_j|\star$

Proving para-NP-hardness



Proving para-*NP*-hardness



- [Flum, Grohe 1999] $(\mathcal{P}, k)$ is para-*NP*-complete iff $\mathcal{P}$ is *NP*-complete for some fixed value of k.

$1|chains(\ell_{i,j}^{ex}), p_j = 1, r_j, d_j|\star$, parameter $\mu = 2$

- Reduction from 3-COLORING: $G = (V, E), n = |V|, m = |E|$

$1|chains(\ell_{i,j}^{ex}), p_j = 1, r_j, d_j|\star$, parameter $\mu = 2$

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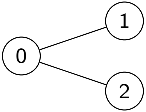
Example:



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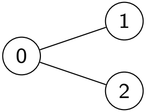
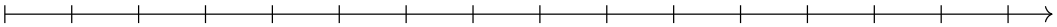
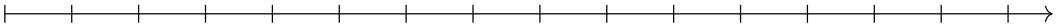


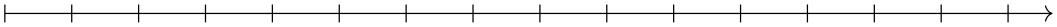
Vertex chain C_0



Vertex chain C_1



Vertex chain C_2



$1|chains(\ell^{ex}_{i,j}), p_j = 1, r_j, d_j|\star$, parameter $\mu = 2$

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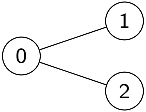
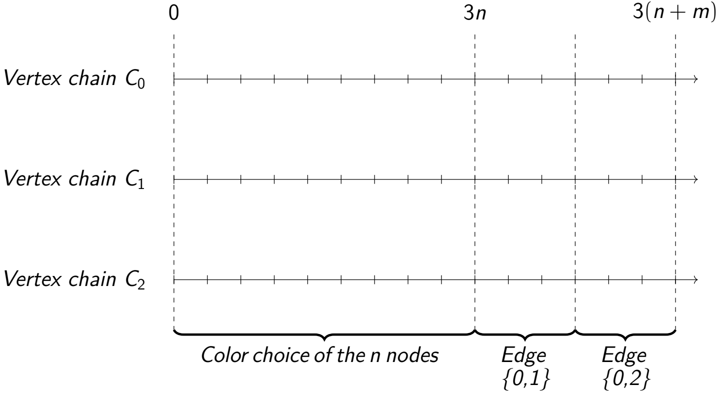
Example:





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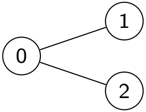

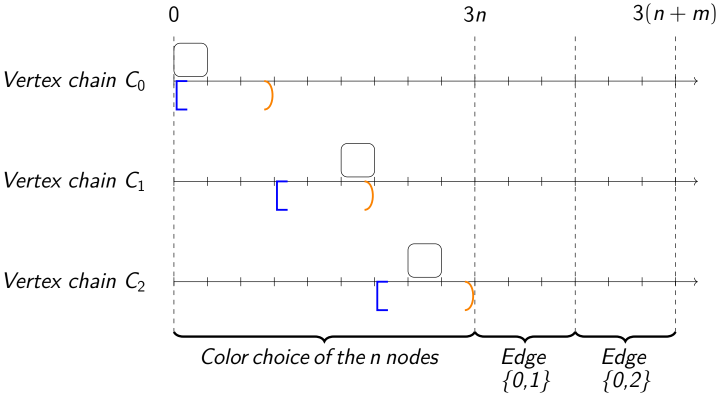

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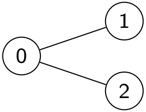

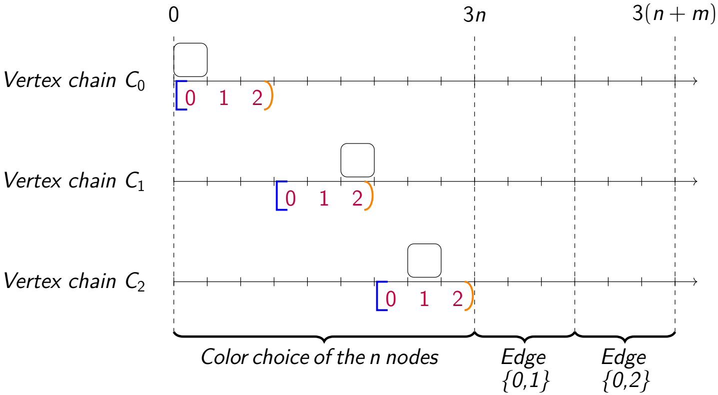

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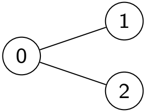

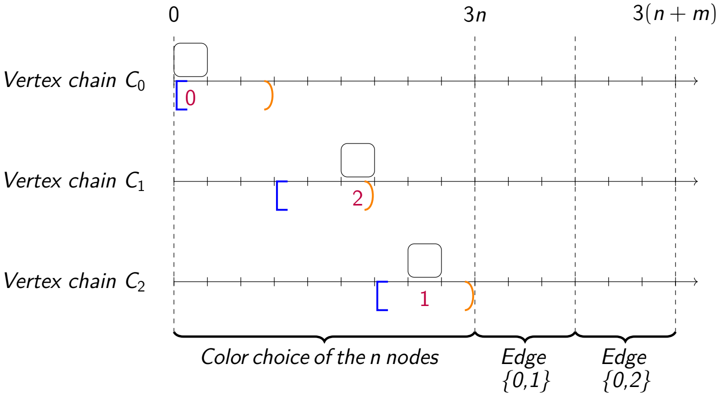

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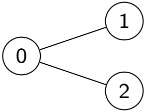

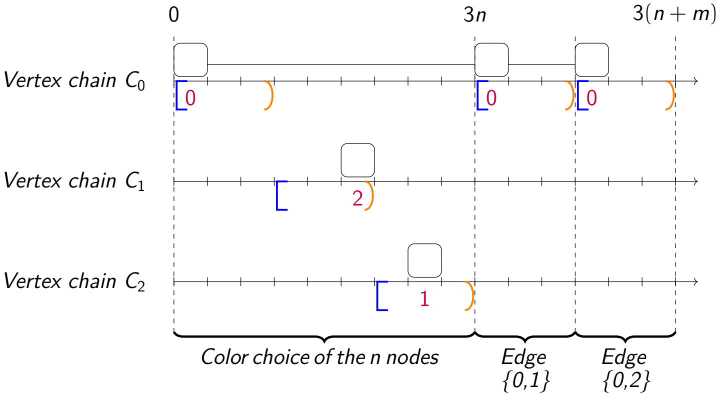

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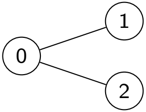

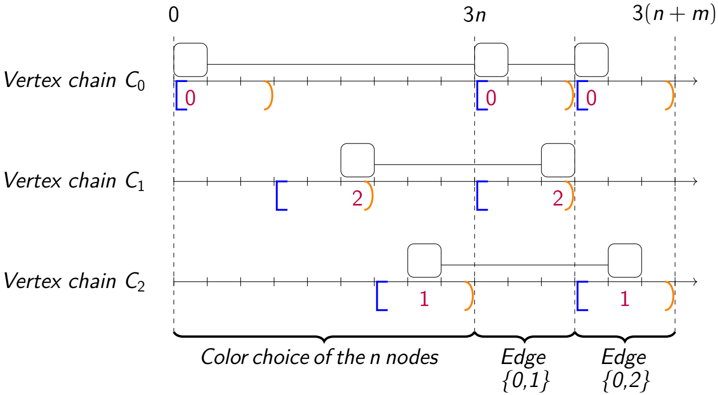

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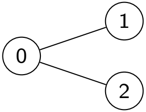

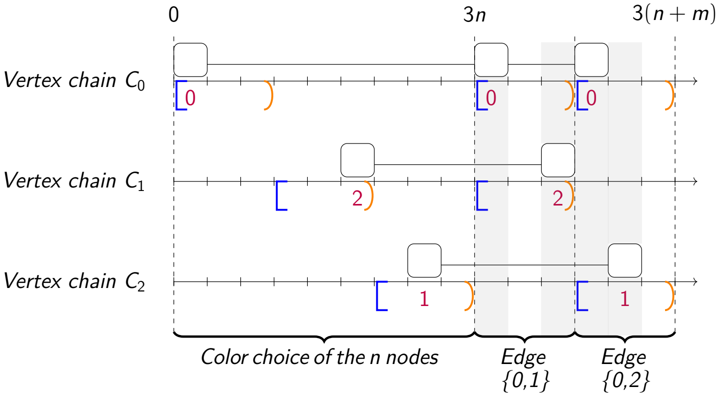

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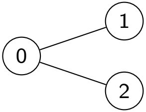
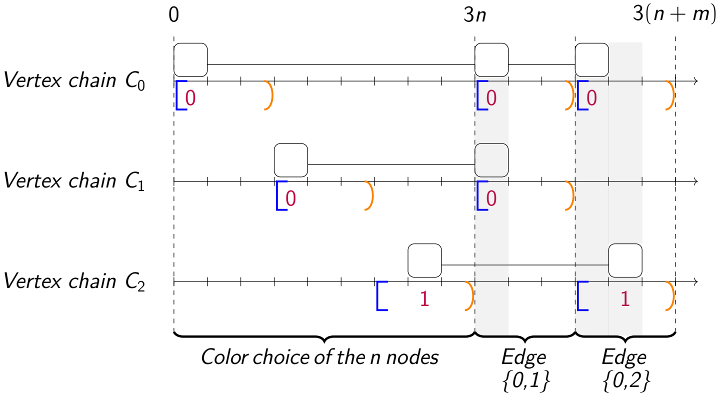
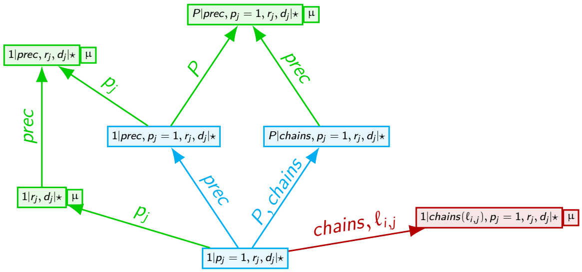
Example:





Parameter μ


$P|prec, p_j = 1, r_j, d_j|\star$ μ
$1|prec, r_j, d_j|\star$ μ
P
$prec$
p_j
$prec$
$1|prec, p_j = 1, r_j, d_j|\star$
$P|chains, p_j = 1, r_j, d_j|\star$
$1|r_j, d_j|\star$ μ
$prec$
$P, chains$
p_j
$chains, \ell_{i,j}$
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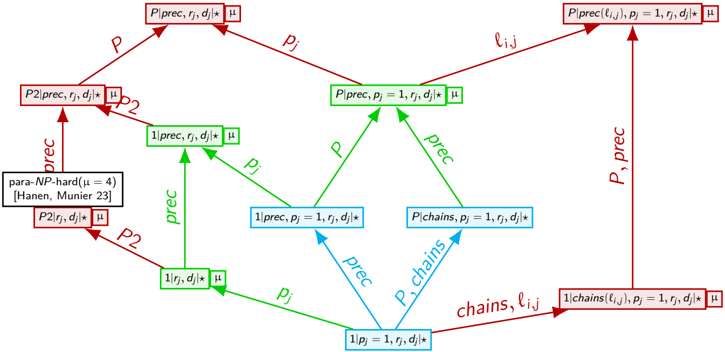
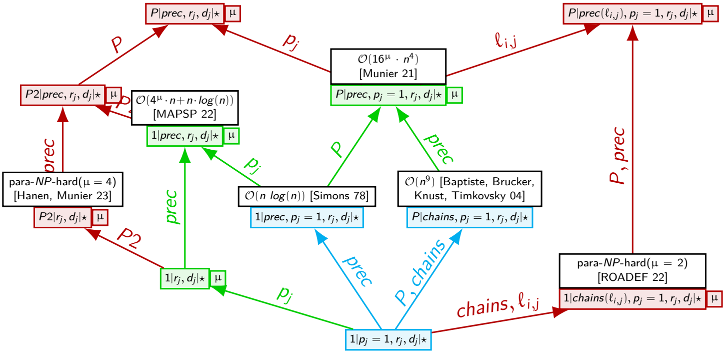
$1|p_j = 1, r_j, d_j|\star$

Parameter μ

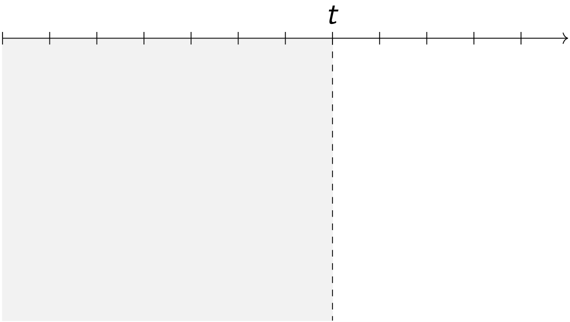

$P|prec, r_j, d_j|\star$ μ
$P|prec(\ell_{i,j}), p_j = 1, r_j, d_j|\star$ μ
P
p_j
$\ell_{i,j}$
$P2|prec, r_j, d_j|\star$ μ
$P|prec, p_j = 1, r_j, d_j|\star$ μ
$P2$
$1|prec, r_j, d_j|\star$ μ
prec
p_j
P
prec
$P, prec$
para-NP-hard($\mu = 4$)
[Hanen, Munier 23]
$P2|r_j, d_j|\star$ μ
prec
$1|prec, p_j = 1, r_j, d_j|\star$
$P|chains, p_j = 1, r_j, d_j|\star$
$P2$
prec
P, chains
$1|r_j, d_j|\star$ μ
p_j
chains, $\ell_{i,j}$
$1|chains(\ell_{i,j}), p_j = 1, r_j, d_j|\star$ μ
$1|p_j = 1, r_j, d_j|\star$

Parameter μ


$P|prec, r_j, d_j|\star$ μ
$P|prec(\ell_{i,j}), p_j = 1, r_j, d_j|\star$ μ
$\mathcal{O}(16^{\mu} \cdot n^4)$ [Munier 21]
$P|prec, p_j = 1, r_j, d_j|\star$ μ
$P2|prec, r_j, d_j|\star$ μ
$\mathcal{O}(4^{\mu} \cdot n + n \cdot log(n))$ [MAPSP 22]
$1|prec, r_j, d_j|\star$ μ
para-NP-hard($\mu = 4$) [Hanen, Munier 23]
$P2|r_j, d_j|\star$ μ
$\mathcal{O}(n\ log(n))$ [Simons 78]
$1|prec, p_j = 1, r_j, d_j|\star$
$\mathcal{O}(n^9)$ [Baptiste, Brucker, Knust, Timkovsky 04]
$P|chains, p_j = 1, r_j, d_j|\star$
para-NP-hard($\mu = 2$) [ROADEF 22]
$1|chains(\ell_{i,j}), p_j = 1, r_j, d_j|\star$ μ
$1|r_j, d_j|\star$ μ
$1|p_j = 1, r_j, d_j|\star$
P
p_j
$\ell_{i,j}$
$prec$
$P, prec$
$P2$
$P, chains$
$chains, \ell_{i,j}$

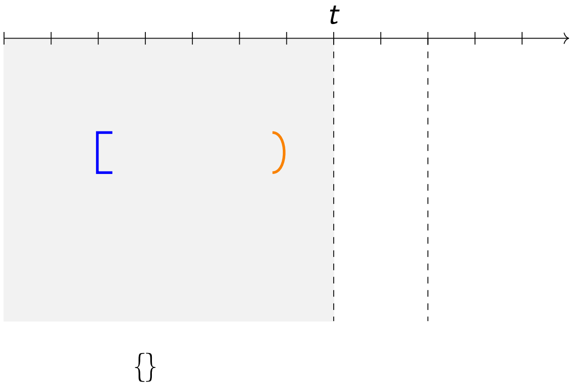
Scheduling
Problem definition
Motivation
Parameterized Complexity
General
Pathwidth
Pathwidth & maximum delay
Further work

- Dynamic programming algorithm on active schedules

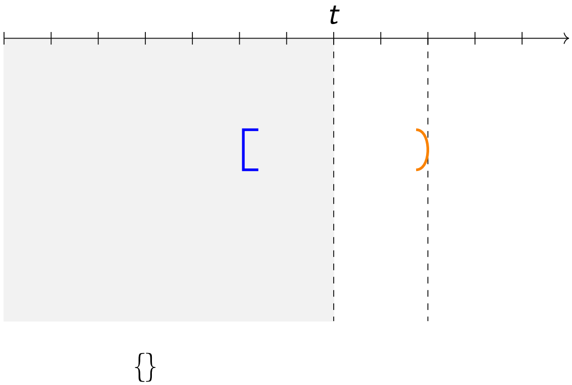


$P|prec(\ell_{i,j}), p_j = 1, r_j, d_j|\star$ is FPT parameterized by $\mu + \ell_{max}$

- Dynamic programming algorithm on active schedules

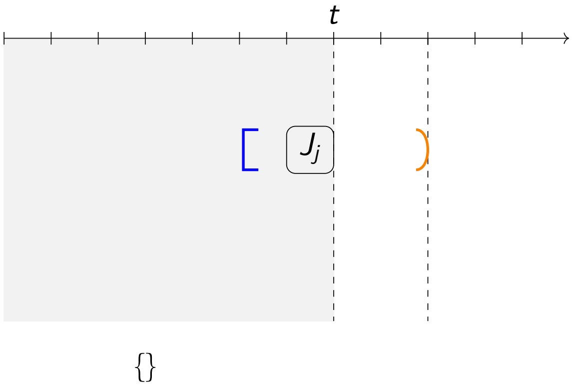


- Dynamic programming algorithm on active schedules



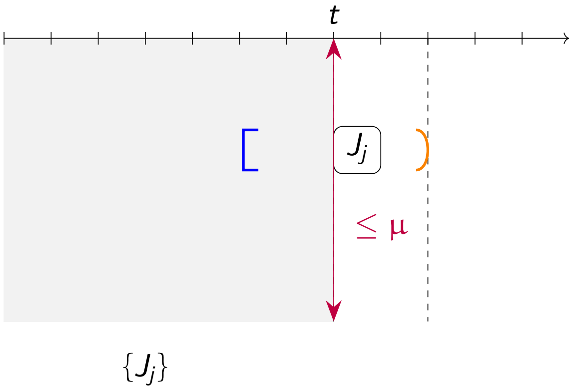
$P|prec(\ell_{i,j}), p_j = 1, r_j, d_j|\star$ is FPT parameterized by $\mu + \ell_{max}$

- Dynamic programming algorithm on active schedules



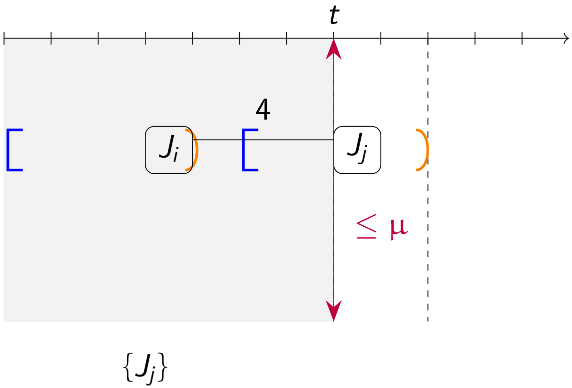
$P|prec(\ell_{i,j}), p_j = 1, r_j, d_j|\star$ is FPT parameterized by $\mu + \ell_{max}$

- Dynamic programming algorithm on active schedules



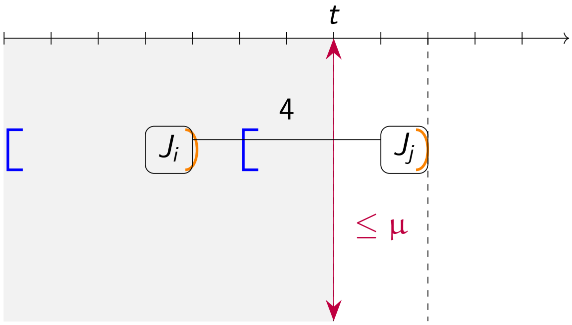
$P|prec(\ell_{i,j}), p_j = 1, r_j, d_j|\star$ is FPT parameterized by $\mu + \ell_{max}$

- Dynamic programming algorithm on active schedules



$P|prec(\ell_{i,j}), p_j = 1, r_j, d_j|\star$ is FPT parameterized by $\mu + \ell_{max}$

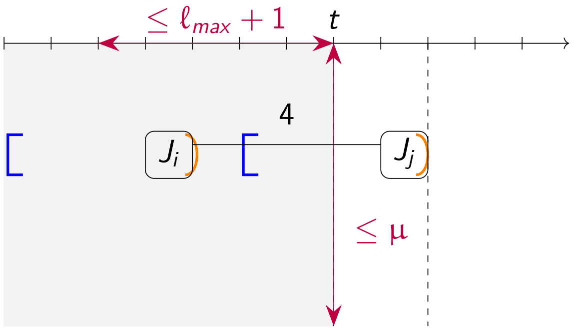
- Dynamic programming algorithm on active schedules



$\{J_j, J_i\}$

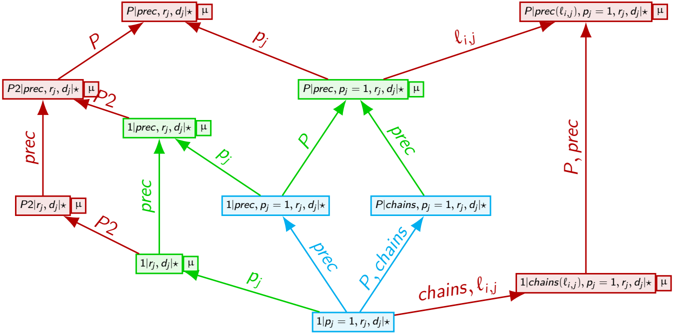
$P|prec(\ell_{i,j}), p_j = 1, r_j, d_j|\star$ is FPT parameterized by $\mu + \ell_{max}$

- Dynamic programming algorithm on active schedules

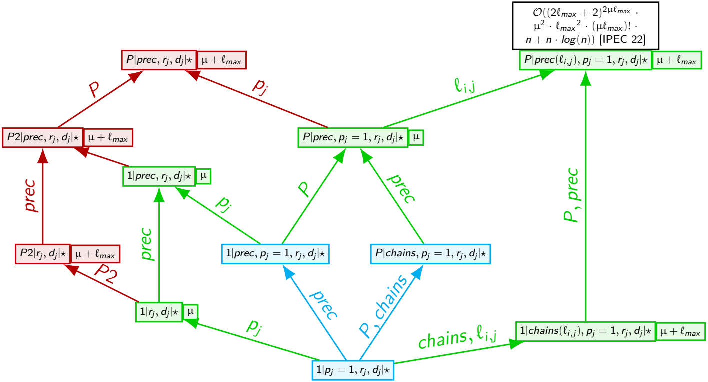


$$|\{J_j, J_i\}| = \mathcal{O}(\mu \times \ell_{max})$$

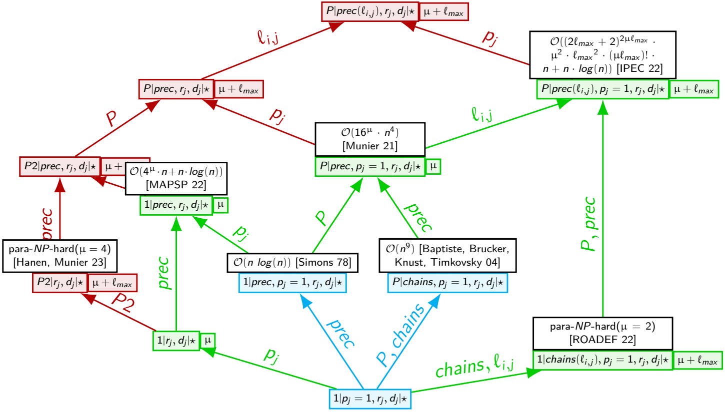
Parameter $\mu + \ell_{max}$



Parameter $\mu + \ell_{max}$



Parameter $\mu + \ell_{max}$



Further work

- *FPT* approximation results

Further work

- *FPT* approximation results
- New temporal-based structural parameters

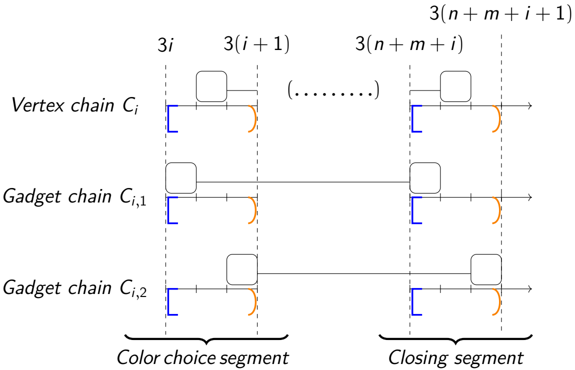
Further work

- *FPT* approximation results
- New temporal-based structural parameters
- Find relations between parameters

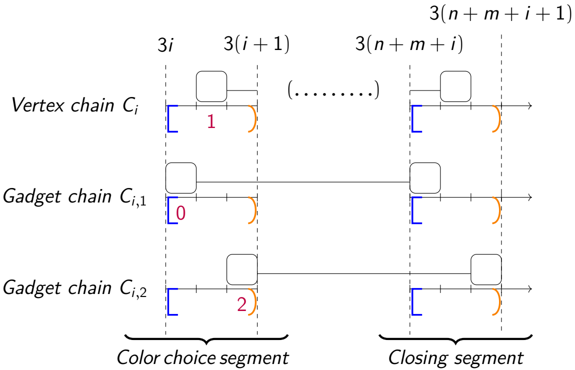
Further work

- *FPT* approximation results
- New temporal-based structural parameters
- Find relations between parameters
  - Parallel machines: $m \leq \mu$
  - m: $P|prec, p_j = 1|C_{max}$ is *XNLP*-hard.
  - μ: $P|prec, p_j = 1, r_j, d_j|C_{max}$ is *FPT*.

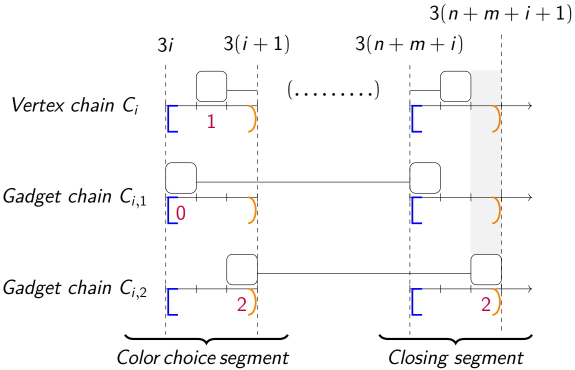
$1|chains(\ell_{i,j}^{min}), p_j = 1, r_j, d_j|\star$, parameter $\mu = 3$



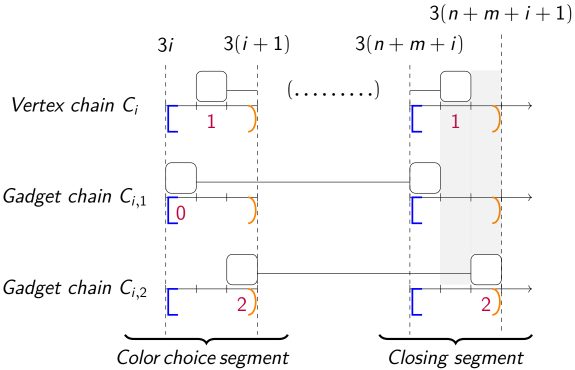
$1|chains(\ell_{i,j}^{min}), p_j = 1, r_j, d_j|\star$, parameter $\mu = 3$



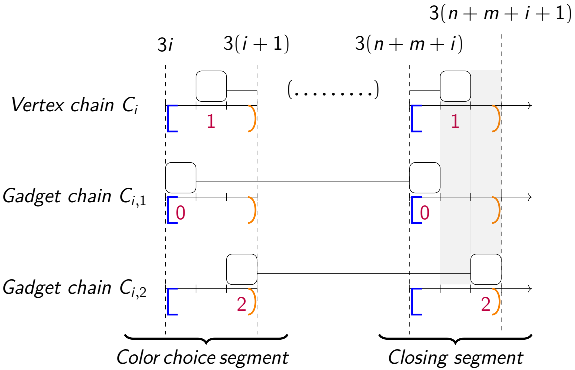
$1|chains(\ell_{i,j}^{min}), p_j = 1, r_j, d_j|\star$, parameter $\mu = 3$



$1|chains(\ell_{i,j}^{min}), p_j = 1, r_j, d_j|\star$, parameter $\mu = 3$



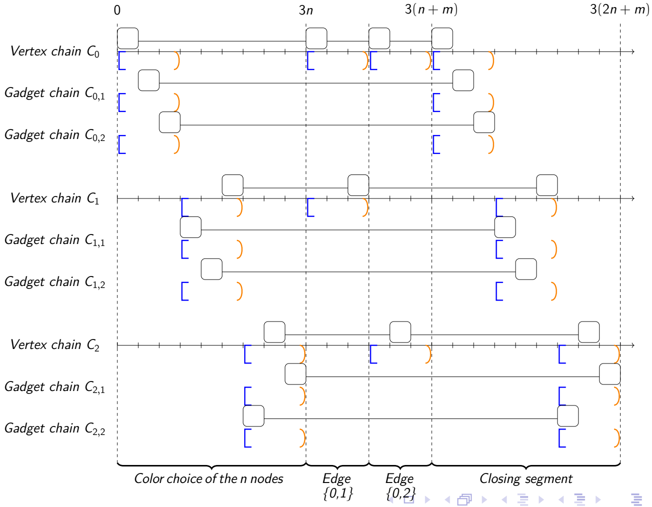
$1|chains(\ell_{i,j}^{min}), p_j = 1, r_j, d_j|\star$, parameter $\mu = 3$



$1|chains(\ell_{i,j}^{min}), p_j = 1, r_j, d_j|\star$, pathwidth 3 (cont.)

- Reduction from 3-COLORING: $G = (V, E), n = |V|, m = |E|$

Example: $V = \{0, 1, 2\}, E = (\{0, 1\}, \{0, 2\})$



$1|chains(\ell_{i,j}^{min}), p_j = 1, r_j, d_j|\star$, pathwidth 3 (cont.)

- Reduction from 3-COLORING: $G = (V, E)$, $n = |V|$, $m = |E|$

Example: $V = \{0, 1, 2\}$, $E = (\{0, 1\}, \{0, 2\})$