Scheduling meets parameterized complexity

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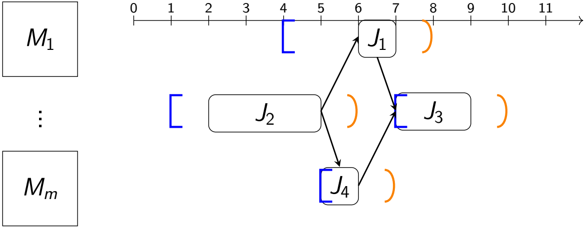
Parallel machine scheduling



$$P|prec, r_j, d_j|\star$$

Exploring complexity frontiers

From poly-time solvable problem $1|p_j = 1, r_j, d_j|\star$

Exploring complexity frontiers

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- $1 \rightarrow P$

Exploring complexity frontiers

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- $1 \longrightarrow P$
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- $p_j = 1 \longrightarrow$ any p_j
- Add precedence relations *prec*

Exploring complexity frontiers

From poly-time solvable problem $1|p_j = 1, r_j, d_j|\star$

- $1 \longrightarrow P$
- $p_j = 1 \longrightarrow$ any p_j
- Add precedence relations *prec*
- Add precedence delays $\ell_{i,j}$

Exploring complexity frontiers

From poly-time solvable problem $1|p_j = 1, r_j, d_j|\star$

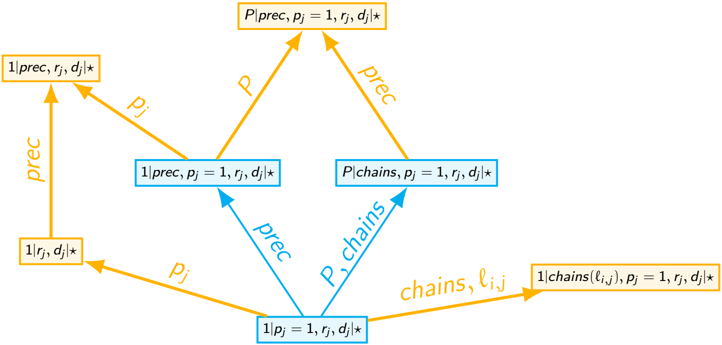
- $1 \longrightarrow P$
- $p_j = 1 \longrightarrow$ any p_j
- Add precedence relations *prec*
- Add precedence delays $\ell_{i,j}$

$1|r_j, d_j|\star$ is strongly NP-hard. [Brucker, Kan, Lenstra 1977]

Classical complexity theory

$\boxed{1|p_j = 1, r_j, d_j|\star}$

Classical complexity theory



Going beyond NP-hardness

- P $\subseteq$ NP
- P: deterministic time $poly(n)$
- NP: nondeterministic time $poly(n)$

Going beyond NP-hardness

- P $\subseteq$ NP
  - FPT $\subseteq$ para-NP, parameter k
- P: deterministic time $poly(n)$
  - FPT: deterministic time $f(k) \times poly(n)$
- NP: nondeterministic time $poly(n)$
  - para-NP: nondeterministic time $f(k) \times poly(n)$

Going beyond NP-hardness

- P $\subseteq$ NP
  - FPT $\subseteq$ para-NP, parameter k
- P: deterministic time $poly(n)$
  - FPT: deterministic time $f(k) \times poly(n)$
- NP: nondeterministic time $poly(n)$
  - para-NP: nondeterministic time $f(k) \times poly(n)$
- FPT $\subseteq W[1] \subseteq W[2] \subseteq \ldots \subseteq$ para-NP.
  - $W[1] - hard =$ "unlikely to be FPT".

- [Bodlaender, Fellows 1995]
  - #machines m: $P|prec, p_j = 1|C_{max}$ is W[2]-hard.

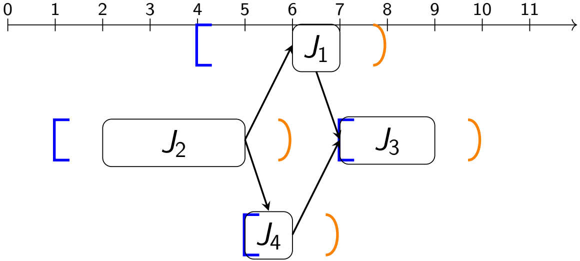
Parameterized complexity in machine scheduling

- [Bodlaender, Fellows 1995]
  - #machines m: $P|prec, p_j = 1|C_{max}$ is W[2]-hard.
- [Mnich, Wiese 2015]
  - Maximum processing time p_{max}: $P||C_{max} \in$ FPT.

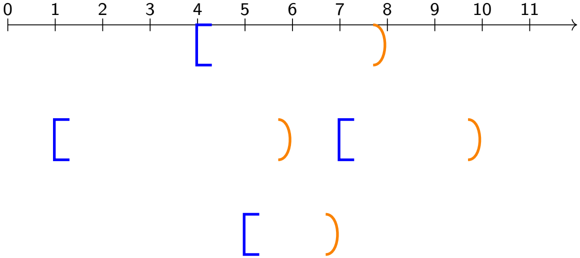
Parameterized complexity in machine scheduling

- [Bodlaender, Fellows 1995]
  - #machines m: $P|prec, p_j = 1|C_{max}$ is W[2]-hard.
- [Mnich, Wiese 2015]
  - Maximum processing time p_{max}: $P||C_{max} \in$ FPT.
- [Hermelin, Karhi, Pinedo, Shabtay 2018]
  - #distinct d_j, #distinct p_j, #distinct w_j: $1||\sum w_j U_j \in$ FPT with two of the three.

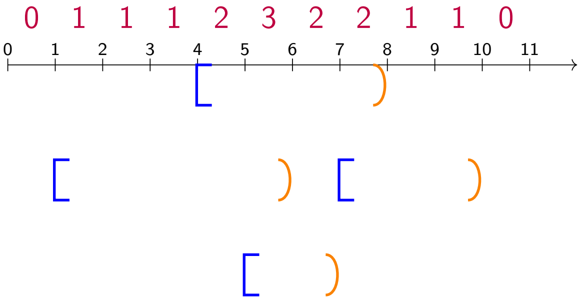
Pathwidth μ



Pathwidth μ

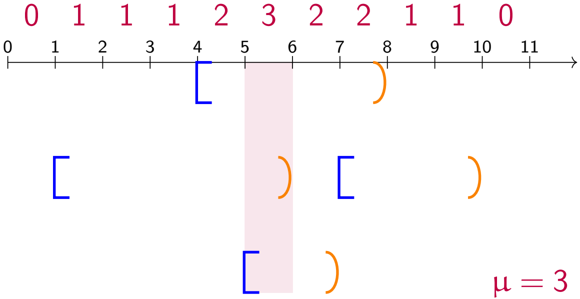


Pathwidth μ



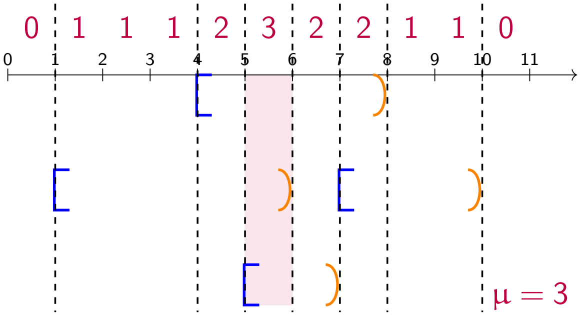
Scheduling
Problem definition
Motivation
Parameterized Complexity
General
Pathwidth
Pathwidth & maximum delay
Conclusion

Pathwidth μ

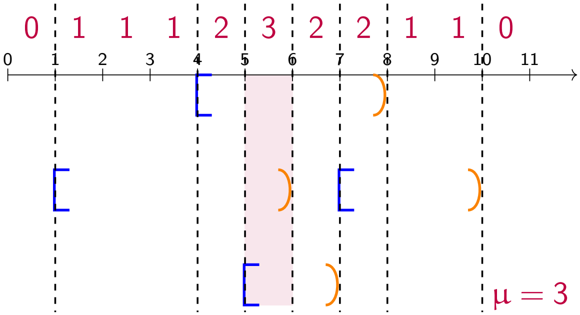


$\mu = 3$

Pathwidth μ

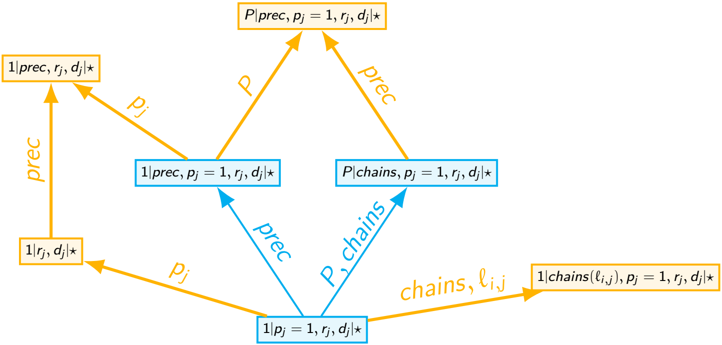


Pathwidth μ

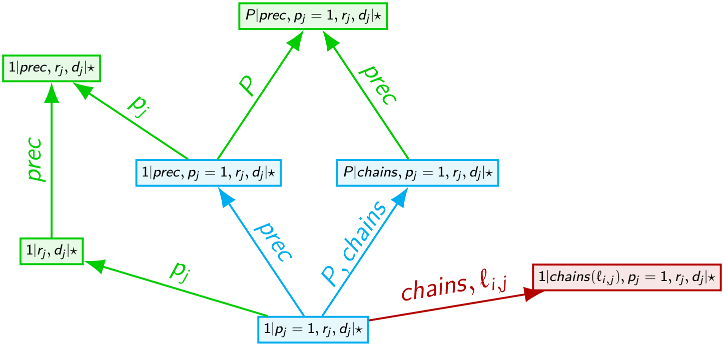


- [Munier 2021] $P|prec, p_j = 1, r_j|L_{max}$ is FPT parameterized by μ.

Parameter μ


$P|prec, p_j = 1, r_j, d_j|\star$
$1|prec, r_j, d_j|\star$
p_j
P
$prec$
$prec$
$1|prec, p_j = 1, r_j, d_j|\star$
$P|chains, p_j = 1, r_j, d_j|\star$
$prec$
$P, chains$
$1|r_j, d_j|\star$
p_j
$chains, \ell_{i,j}$
$1|chains(\ell_{i,j}), p_j = 1, r_j, d_j|\star$
$1|p_j = 1, r_j, d_j|\star$

Parameter μ


$P|prec, p_j = 1, r_j, d_j|\star$
$1|prec, r_j, d_j|\star$
P
$prec$
p_j
$prec$
$1|prec, p_j = 1, r_j, d_j|\star$
$P|chains, p_j = 1, r_j, d_j|\star$
$1|r_j, d_j|\star$
p_j
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$P, chains$
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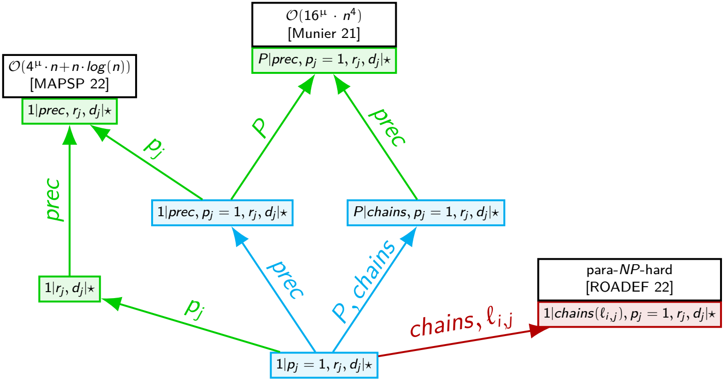
Parameter μ


$\mathcal{O}(16^{\mu} \cdot n^4)$
[Munier 21]
$P|prec, p_j = 1, r_j, d_j|\star$
$\mathcal{O}(4^{\mu} \cdot n + n \cdot log(n))$
[MAPSP 22]
$1|prec, r_j, d_j|\star$
P
$prec$
p_j
$prec$
$1|prec, p_j = 1, r_j, d_j|\star$
$P|chains, p_j = 1, r_j, d_j|\star$
$1|r_j, d_j|\star$
p_j
$prec$
$P, chains$
para-NP-hard
[ROADEF 22]
$1|chains(\ell_{i,j}), p_j = 1, r_j, d_j|\star$
$chains, \ell_{i,j}$
$1|p_j = 1, r_j, d_j|\star$

$1|chains(\ell^{ex}_{i,j}), p_j = 1, r_j, d_j|\star$, parameter $\mu = 2$

- Reduction from 3-COLORING: $G = (V, E), n = |V|, m = |E|$

$1|chains(\ell_{i,j}^{ex}), p_j = 1, r_j, d_j|\star$, parameter $\mu = 2$

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Example: $V = \{0, 1, 2\}, E = (\{0, 1\}, \{0, 2\})$

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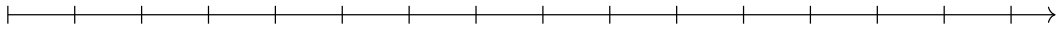
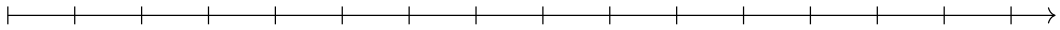
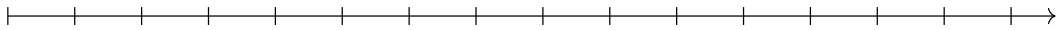
Vertex chain C_0



Vertex chain C_1



Vertex chain C_2



$1|chains(\ell_{i,j}^{ex}), p_j = 1, r_j, d_j|\star$, parameter $\mu = 2$

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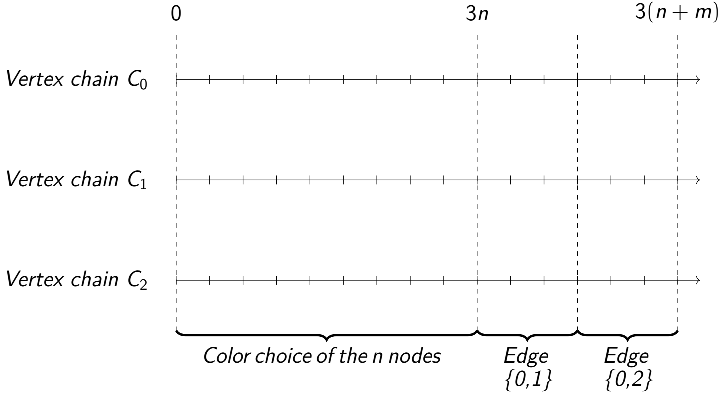

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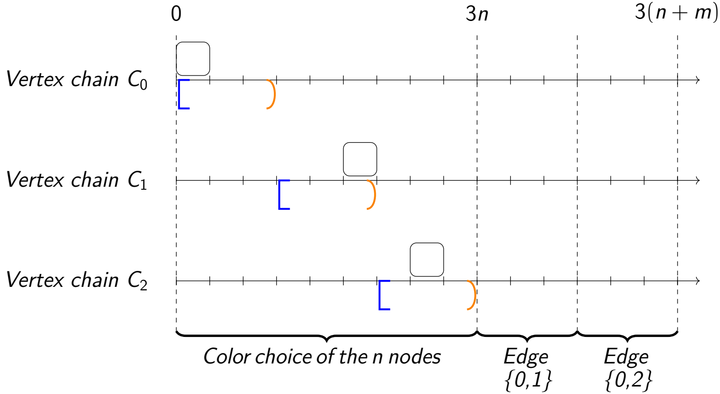

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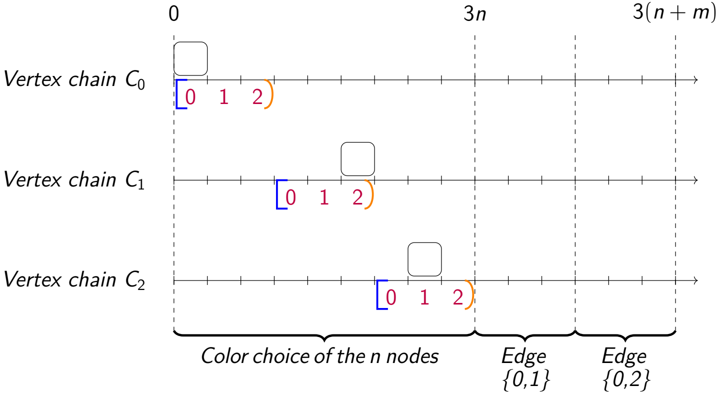

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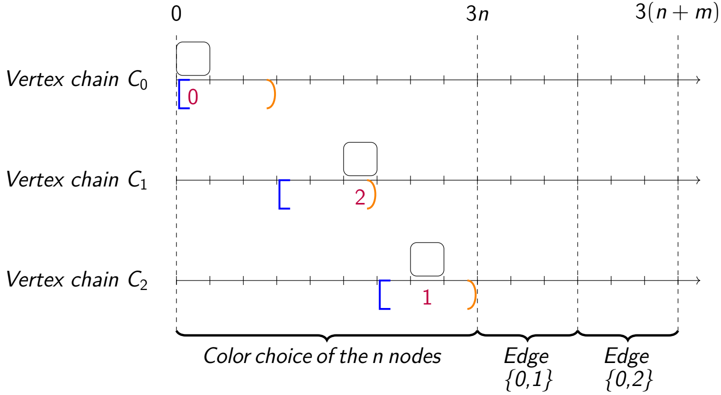

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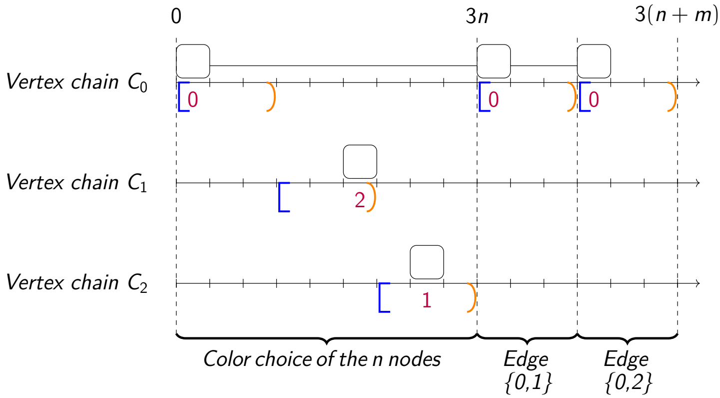

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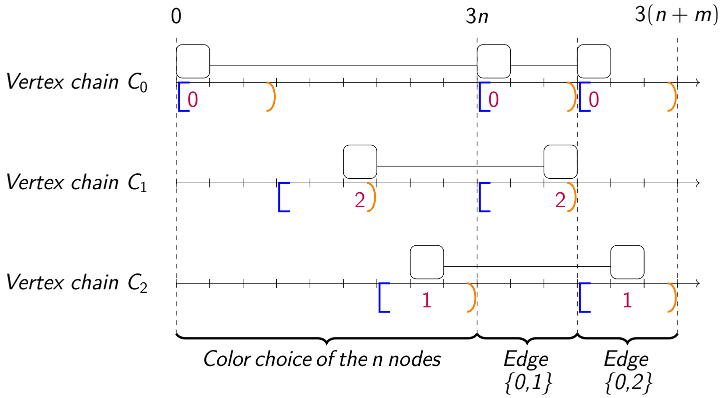

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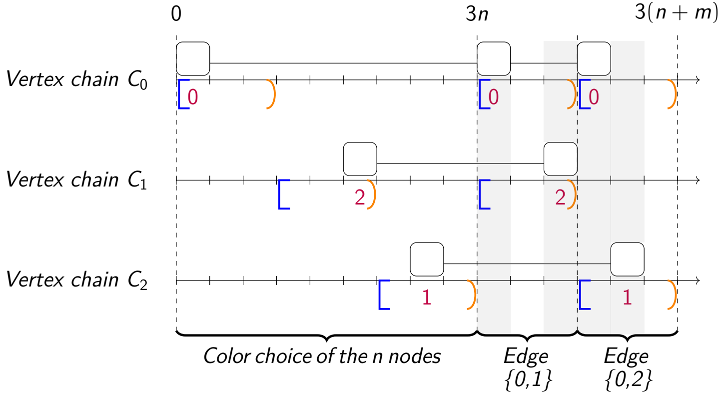
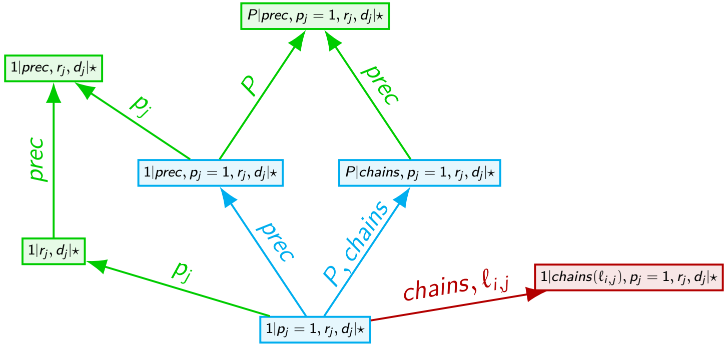
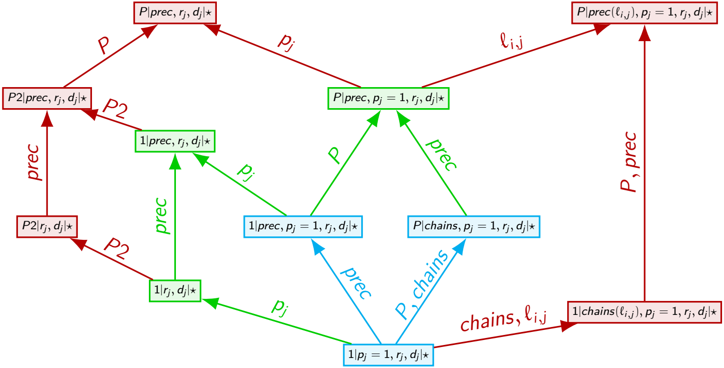


Parameter μ



Parameter µ



$P|prec, r_j, d_j|\star$
$P|prec(\ell_{i,j}), p_j = 1, r_j, d_j|\star$
P
p_j
$\ell_{i,j}$
$P2|prec, r_j, d_j|\star$
$P2$
$P|prec, p_j = 1, r_j, d_j|\star$
$1|prec, r_j, d_j|\star$
$prec$
p_j
P
$prec$
$P, prec$
$P2|r_j, d_j|\star$
$prec$
$1|prec, p_j = 1, r_j, d_j|\star$
$P|chains, p_j = 1, r_j, d_j|\star$
$P2$
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$P, chains$
$1|r_j, d_j|\star$
p_j
$chains, \ell_{i,j}$
$1|chains(\ell_{i,j}), p_j = 1, r_j, d_j|\star$
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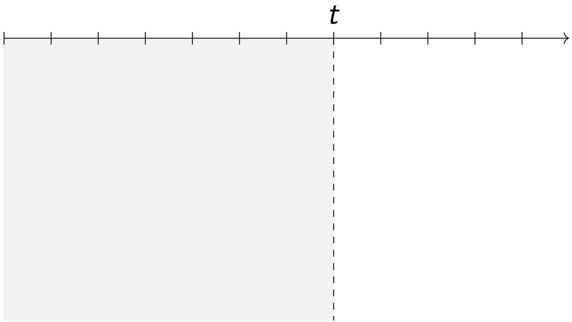
$P|prec(\ell_{i,j}), p_j = 1, r_j, d_j|\star$ is FPT parameterized by $\mu + \ell_{max}$

- Dynamic programming algorithm on active schedules



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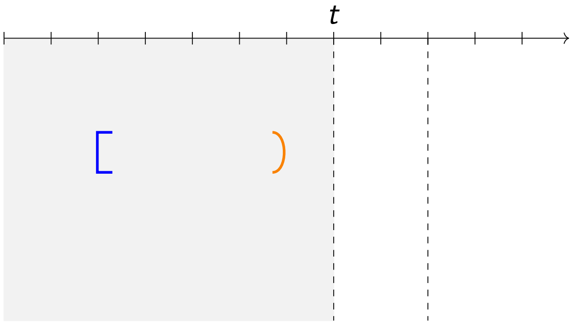

$\{\}$

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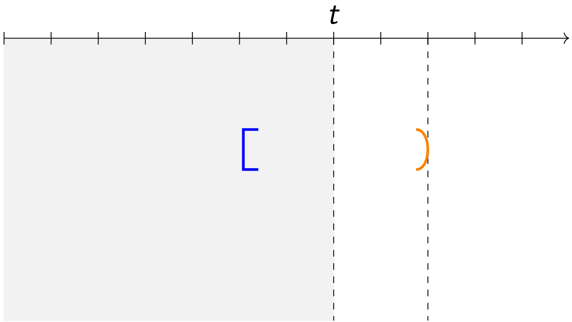

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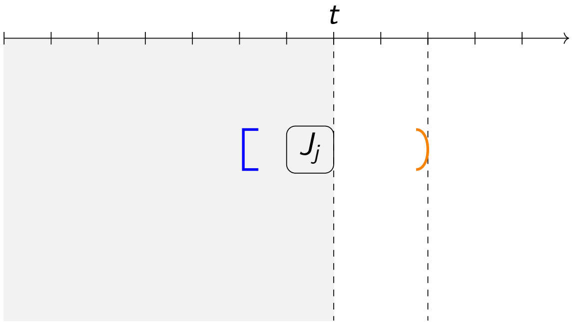

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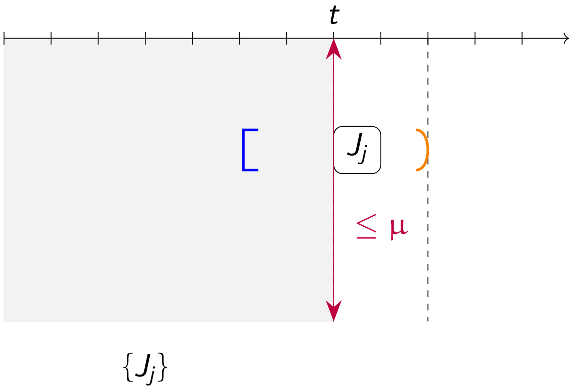

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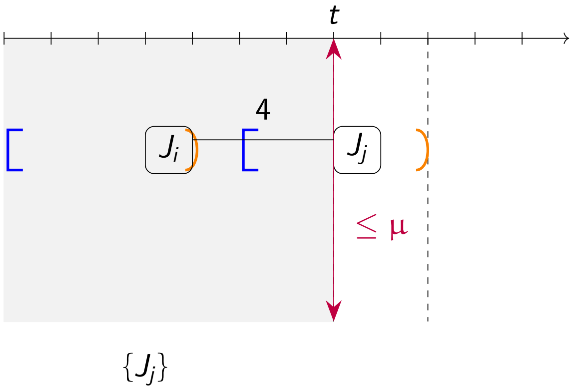

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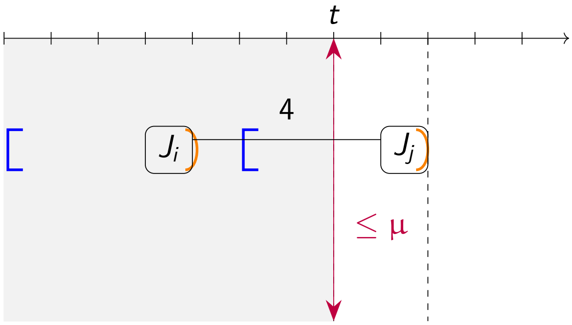
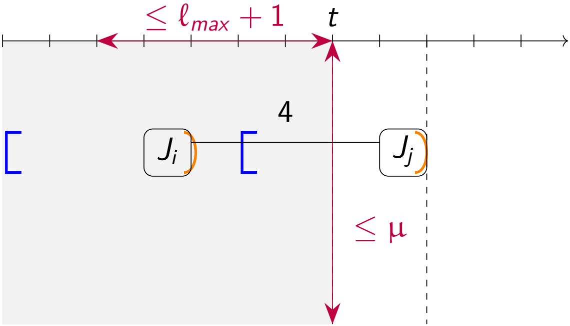


$\{J_j, J_i\}$

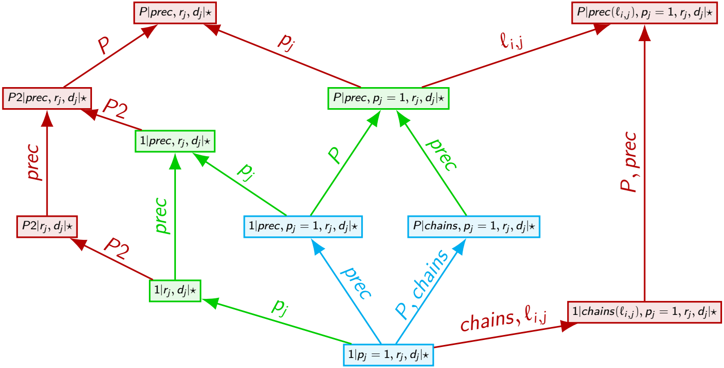
$P|prec(\ell_{i,j}), p_j = 1, r_j, d_j|\star$ is FPT parameterized by $\mu + \ell_{max}$

- Dynamic programming algorithm on active schedules

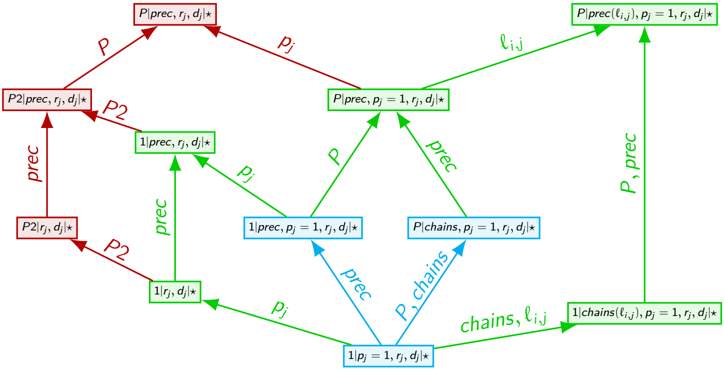


$$|\{J_j, J_i\}| = \mathcal{O}(\mu \times \ell_{max})$$

Parameter $\mu + \ell_{max}$



Parameter $\mu + \ell_{max}$



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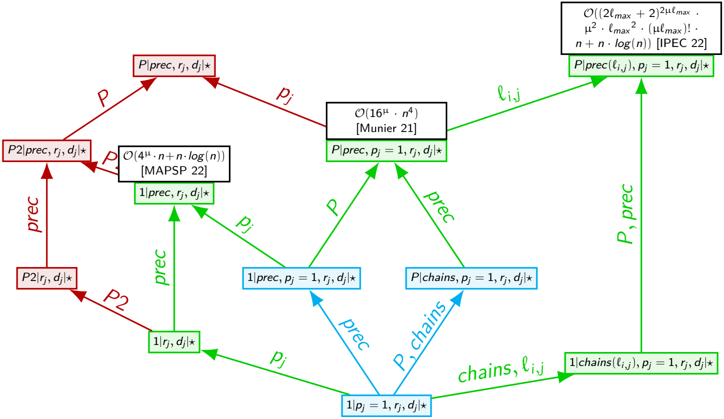
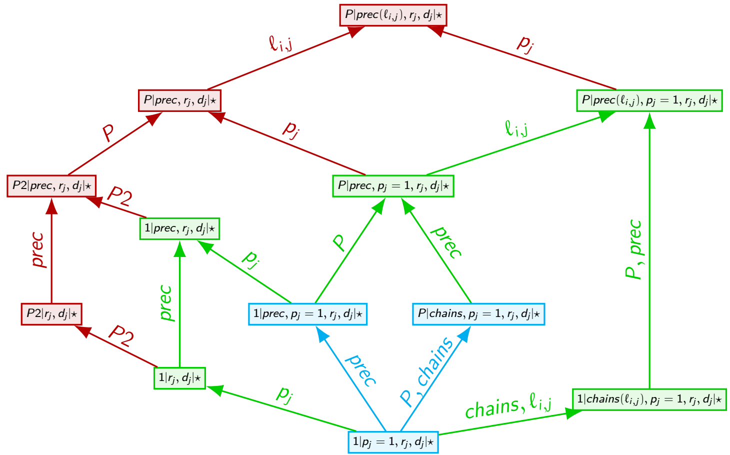


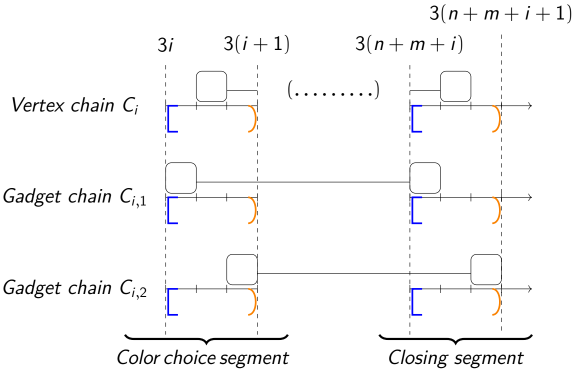
Parameter $\mu + \ell_{max}$



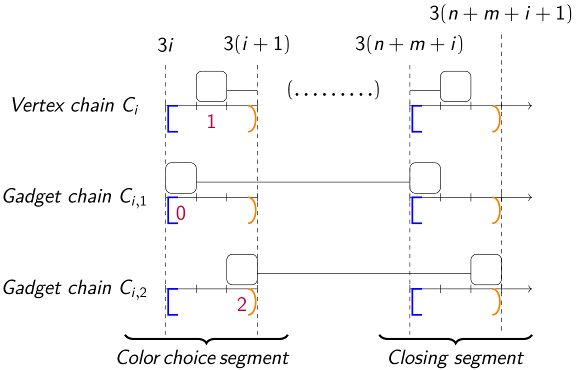
Conclusion

- In-depth classification of scheduling problems (beyond P vs NP).
- More instance families with efficient exact methods.
- Scheduling Zoo [Dürr et al. 2006-Present]

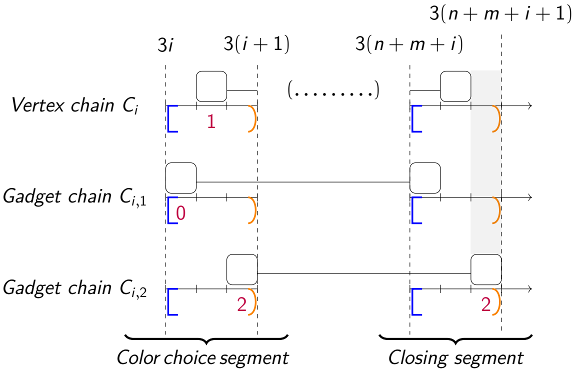
$1|chains(\ell^{min}_{i,j}), p_j = 1, r_j, d_j|\star$, parameter $\mu = 3$



$1|chains(\ell_{i,j}^{min}), p_j = 1, r_j, d_j|\star$, parameter $\mu = 3$



$1|chains(\ell^{min}_{i,j}), p_j = 1, r_j, d_j|\star$, parameter $\mu = 3$



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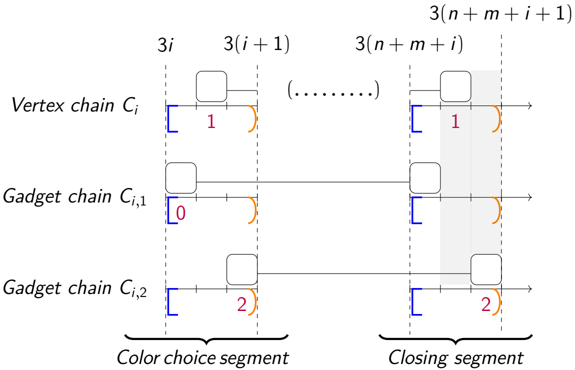

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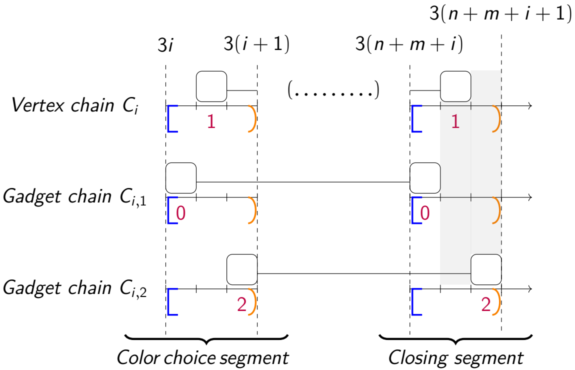
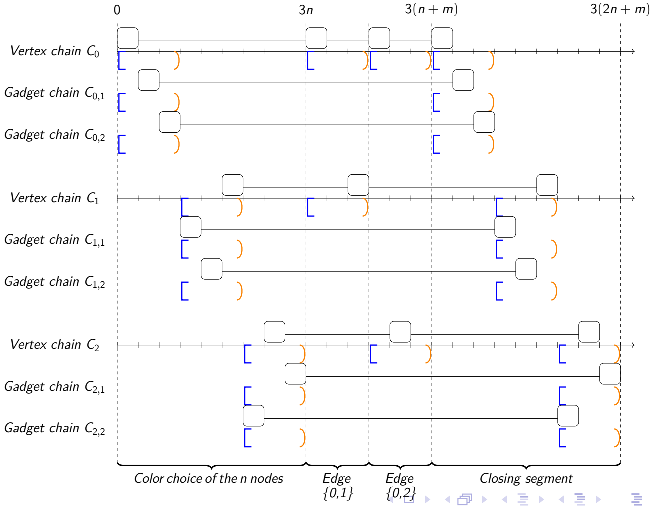


$1|chains(\ell_{i,j}^{min}), p_j = 1, r_j, d_j|\star$, pathwidth 3 (cont.)

- Reduction from 3-COLORING: $G = (V, E)$, $n = |V|$, $m = |E|$

Example: $V = \{0, 1, 2\}$, $E = (\{0, 1\}, \{0, 2\})$



$1|chains(\ell_{i,j}^{min}), p_j = 1, r_j, d_j|\star$, pathwidth 3 (cont.)

- Reduction from 3-COLORING: $G = (V, E)$, $n = |V|$, $m = |E|$

Example: $V = \{0, 1, 2\}$, $E = (\{0, 1\}, \{0, 2\})$