Parameterized Complexity of a Parallel Machine Scheduling Problem

Maher Mallem, Claire Hanen, Alix Munier-Kordon

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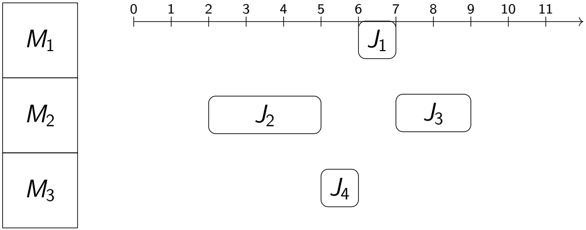
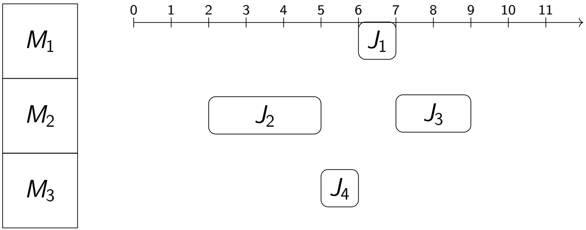
Parallel machine scheduling

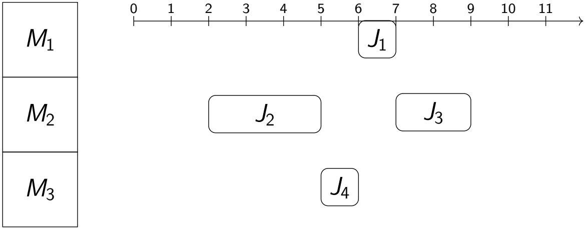


Parallel machine scheduling



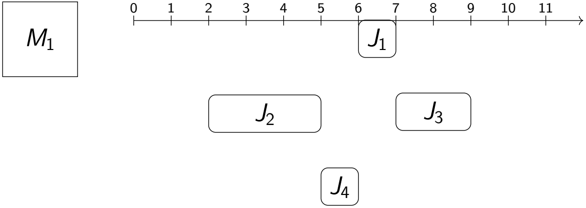
machines | job properties | optimization criterion

Parallel machine scheduling



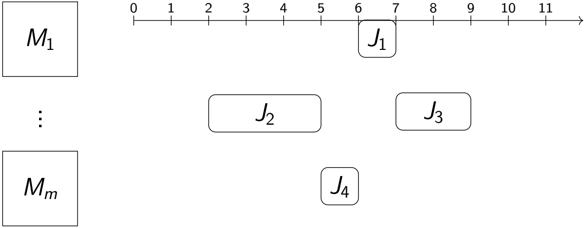
$$P3|.|.$$

Parallel machine scheduling



$1|.|.$

Parallel machine scheduling



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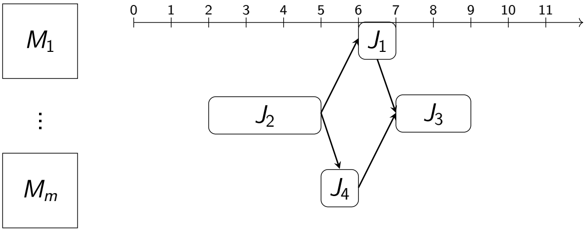
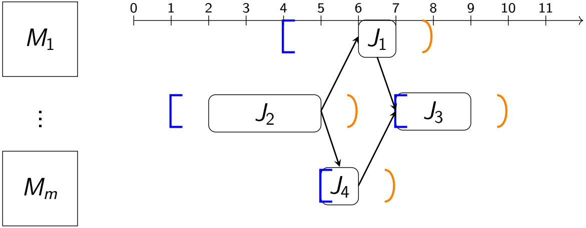
Parallel machine scheduling



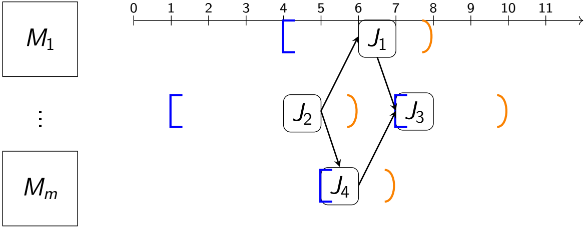
$P|prec|.$

Parallel machine scheduling



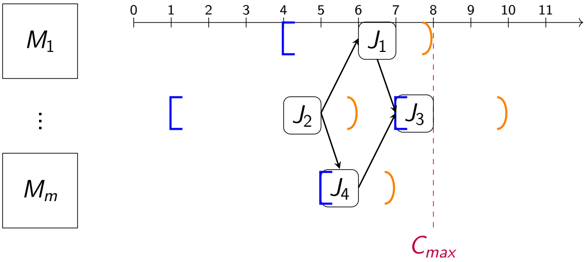
$$P|prec, r_j, d_j|.$$

Parallel machine scheduling



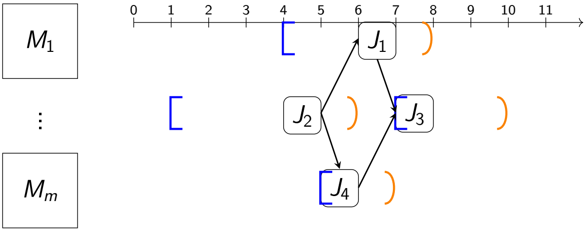
$$P|prec, p_j = 1, r_j, d_j|.$$

Parallel machine scheduling



$$P|prec, p_j = 1, r_j, d_j|C_{max}$$

Parallel machine scheduling



$$P|prec, p_j = 1, r_j, d_j|\star$$

- [Bodlaender, Fellows 1995]
  - #machines m: $P|prec, p_j = 1|C_{max}$ is W[2]-hard.

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- [Mnich, Wiese 2015]
  - Maximum processing time p_{max}: $P||C_{max} \in$ FPT.

Parameterized complexity in machine scheduling

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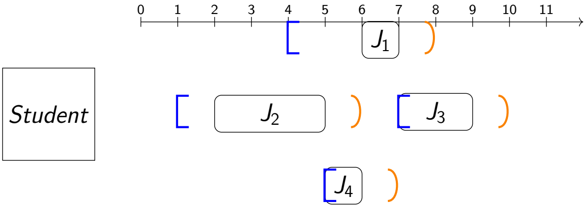
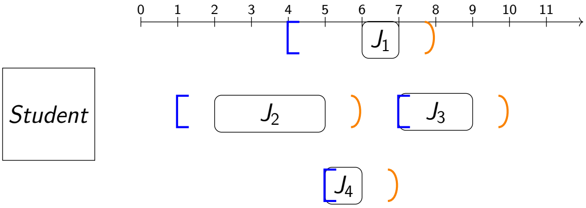
- [Mnich, Wiese 2015]
  - Maximum processing time p_{max}: $P||C_{max} \in$ FPT.
- [Hermelin, Karhi, Pinedo, Shabtay 2018]
  - #distinct d_j, #distinct p_j, #distinct w_j: $1||\sum w_j U_j \in$ FPT with two of the three.

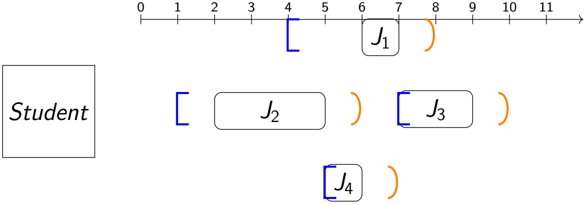
Motivation





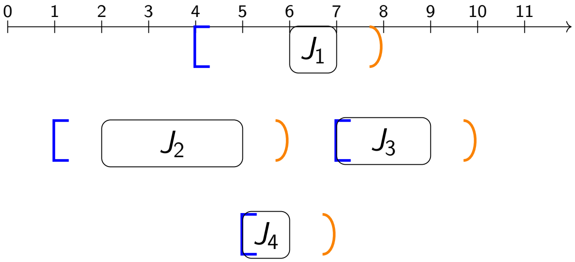
$1|r_j, d_j|\star$

Motivation



$1|r_j, d_j|\star$ is strongly NP-hard.

Pathwidth μ



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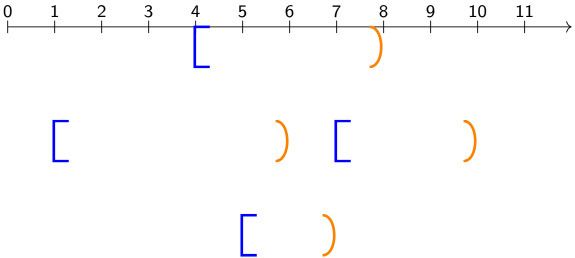

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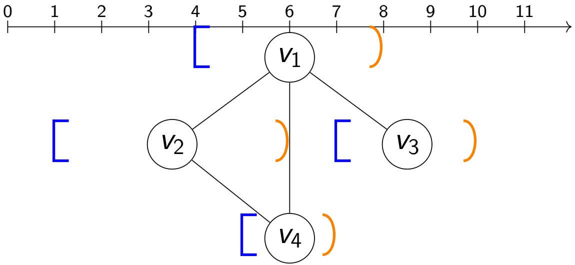

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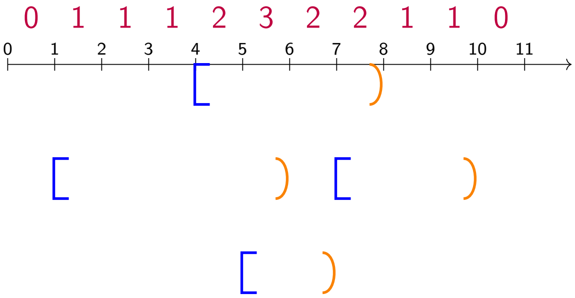

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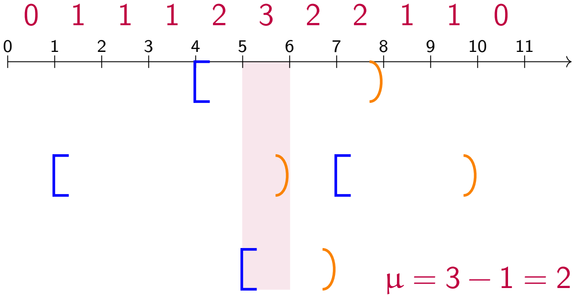
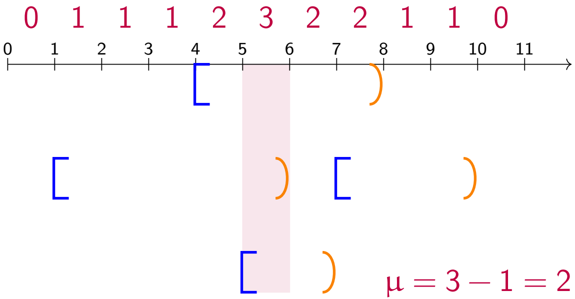


$$\mu = 3 - 1 = 2$$

Pathwidth μ



$\mu = 3 - 1 = 2$

Definition: pathwidth

$$\mu \stackrel{\text{def}}{=} \max_{0 \le t < \max_j d_j} |\{j \mid r_j \le t < d_j\}| - 1$$

Pathwidth μ (cont.)

Recent work:

- [de Weerdt et al. 2020] $1|r_j, s_{jk}, reject, \tilde{d}_j| \sum_{j \notin R} v_j - w_j T_j$ is FPT parameterized by (μ+slack).
- [Munier-Kordon 2021] $P|prec, p_j = 1, r_j|L_{max}$ is FPT parameterized by μ.

Pathwidth vs. precedence delays

- Delays added to precedence relations: what happens?

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  - Exact delay $\ell_{i,j}^{ex}$: $J_i \prec J_j \implies \sigma(j) = \sigma(i) + p_i + \ell_{i,j}^{ex}$
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- Feasibility problem $P|prec(\ell_{i,j}), p_j = 1, r_j, d_j|\star$
  - INPUT: n unit-time jobs J_j with their release date r_j and deadline d_j, m identical machines, a precedence graph $prec$ with delays $\ell_{i,j}$ on its edges.
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- Parameters
  - Pathwidth μ
  - Maximum delay ℓ_{max}

Results

Section	Problem	Parameter	Delays	Result	Proof
2.1	$1\|chains(\ell_{i,j}), p_j = 1, r_j, d_j\|\star$	μ	exact	para-NP-complete	NP-complete when $\mu = 1$
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2.3	$P\|chains(\ell_{i,j}), p_j = 1, r_j, d_j\|\star$	ℓ_{max}	exact	para-NP-complete	NP-complete when $\ell_{max} = 1$
2.4			minimum		
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- Reduction from 3-COLORING: $G = (V, E), n = |V|, m = |E|$

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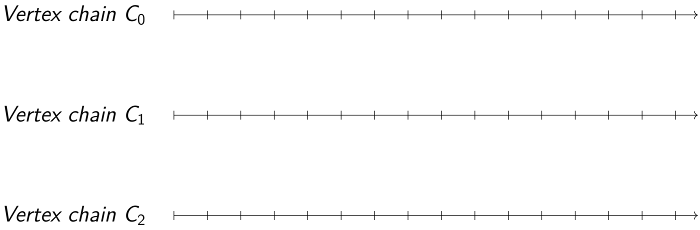

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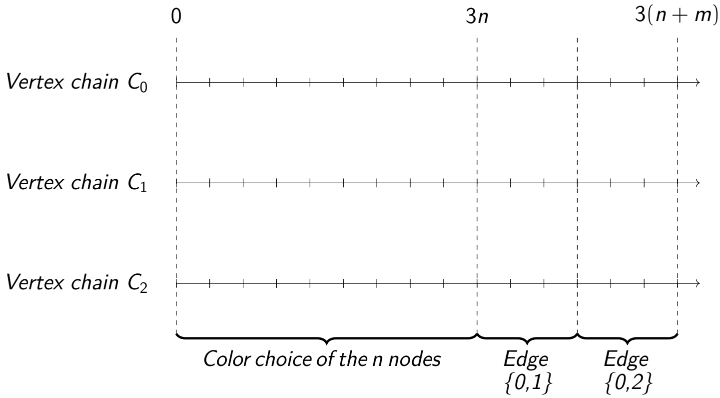

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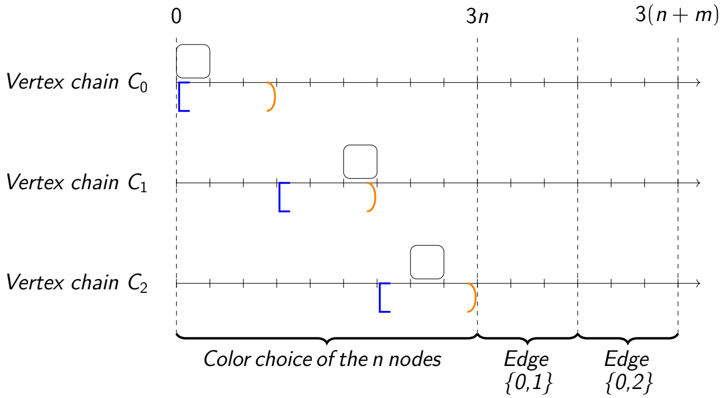

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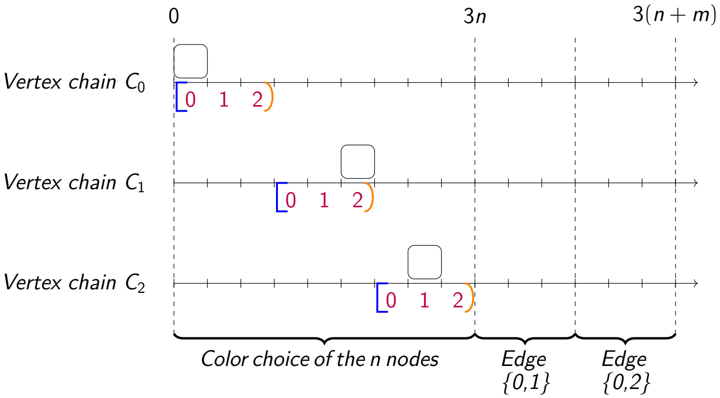

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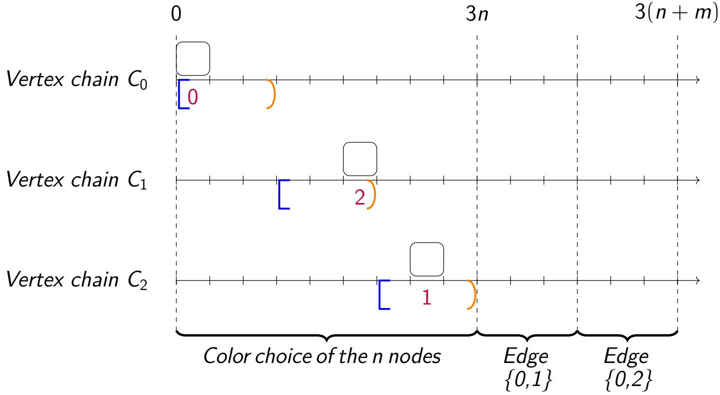

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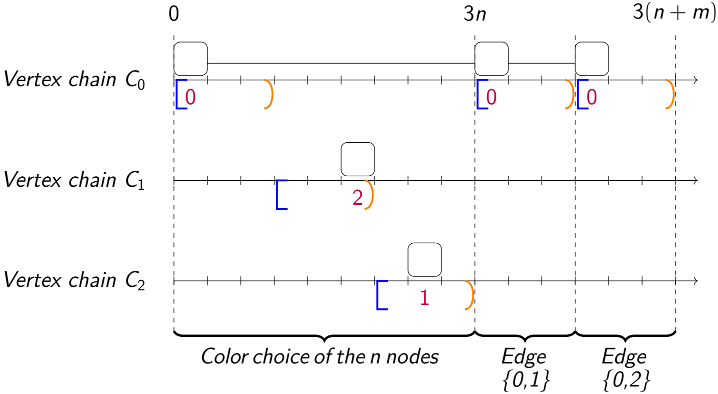

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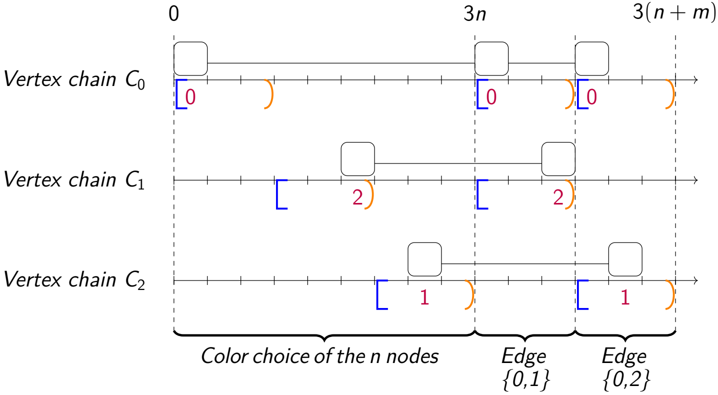

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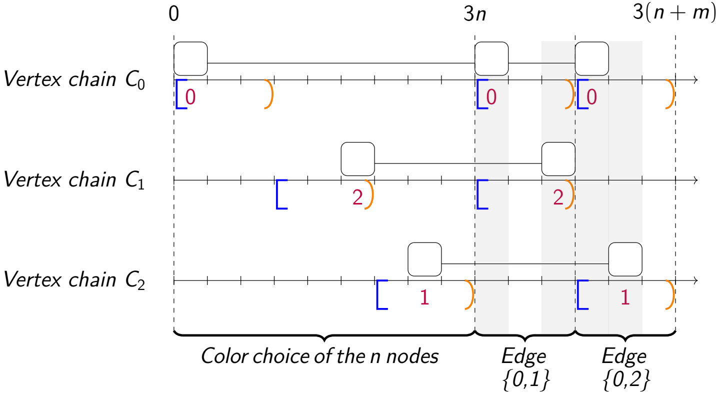

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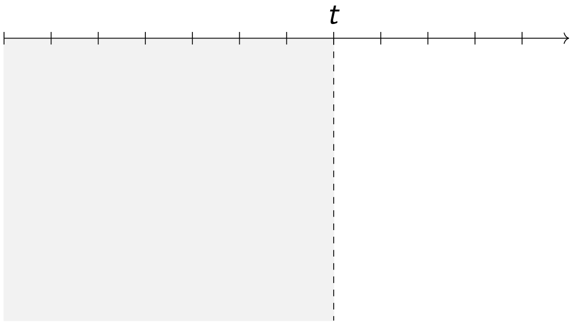

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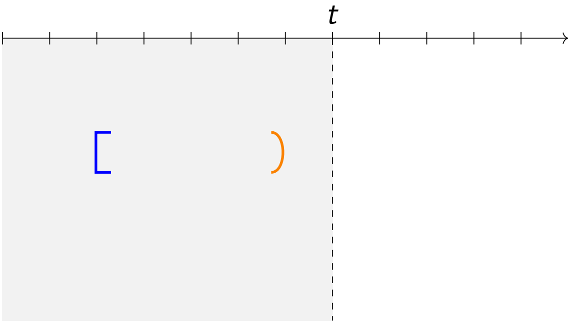

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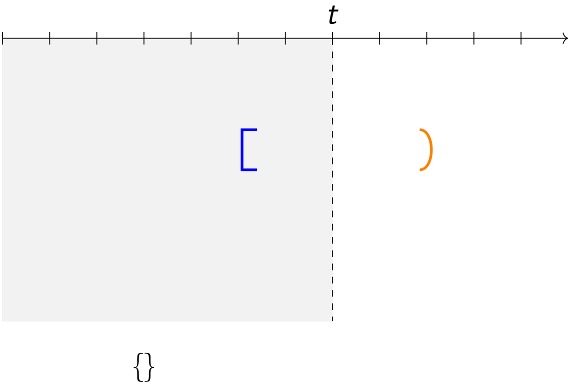

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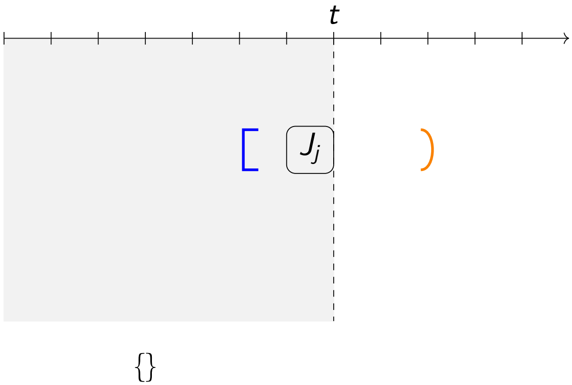

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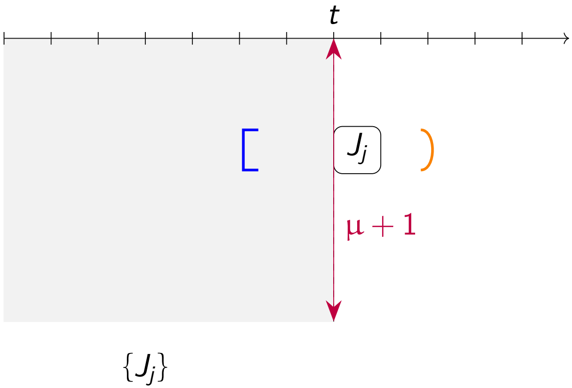

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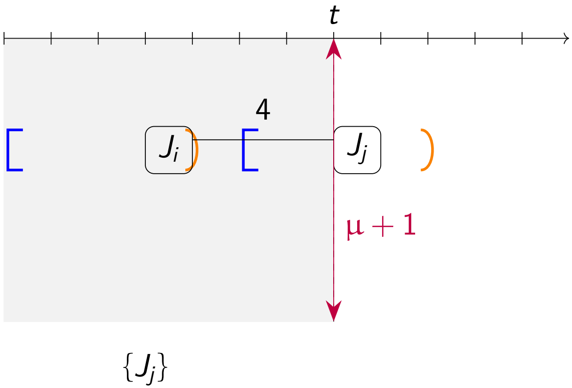

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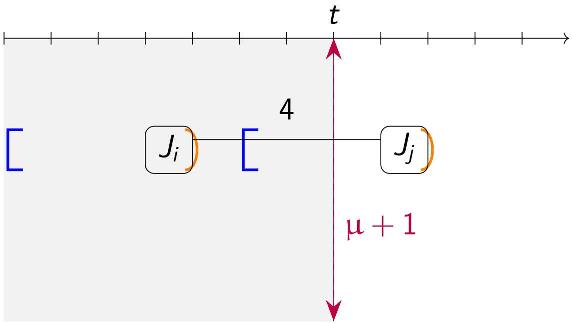

$\{J_j, J_i\}$

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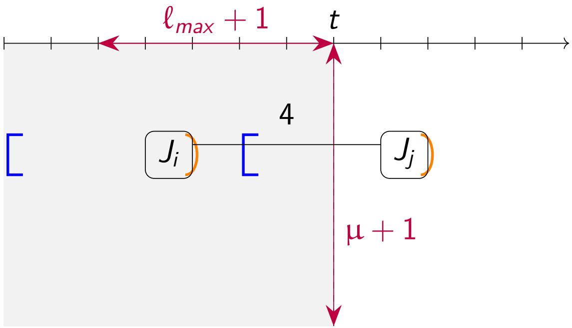

$$|\{J_j, J_i\}| = \mathcal{O}(\mu \times \ell_{max})$$

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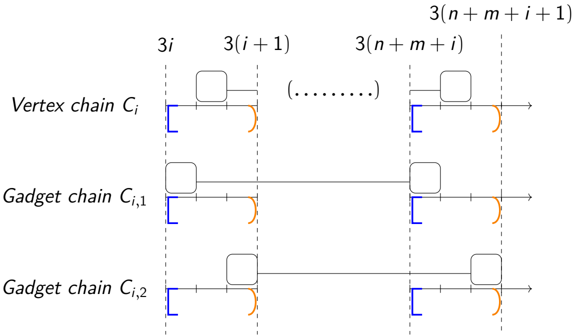
2.3	$P \mid chains(\ell_{i,j}), p_j = 1, r_j, d_j \mid \star$	ℓ_{max}	exact	para-NP-complete	NP-complete when $\ell_{max} = 1$
2.4			minimum		
3.	$P \mid prec(\ell_{i,j}), p_j = 1, r_j, d_j \mid \star$	$\mu + \ell_{max}$	exact	FPT	
			minimum		
/	$P \mid prec(\ell_{i,j}), r_j, d_j \mid \star$	$\mu + \ell_{max} + p_{max}$	exact	?	
			minimum		

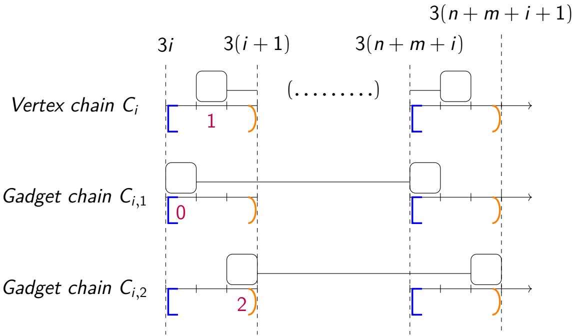
$1|chains(l_{i,j}^{min}), p_j = 1, r_j, d_j|\star$, parameter $\mu = 2$

Section	Problem	Parameter	Delays	Result	Proof
2.1	$1\|chains(l_{i,j}), p_j = 1, r_j, d_j\|\star$	μ	exact	para-NP-complete	NP-complete when $\mu = 1$
2.2			minimum		NP-complete when $\mu = 2$
2.3	$P\|chains(l_{i,j}), p_j = 1, r_j, d_j\|\star$	l_{max}	exact	para-NP-complete	NP-complete when $l_{max} = 1$
2.4			minimum		
3.	$P\|prec(l_{i,j}), p_j = 1, r_j, d_j\|\star$	$\mu + l_{max}$	exact	FPT	
			minimum		

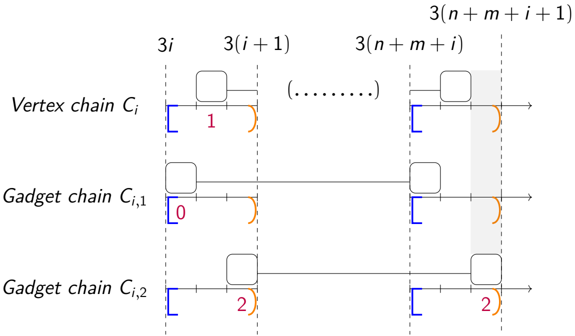
$1|chains(l_{i,j}^{min}), p_j = 1, r_j, d_j|\star$, parameter $\mu = 2$



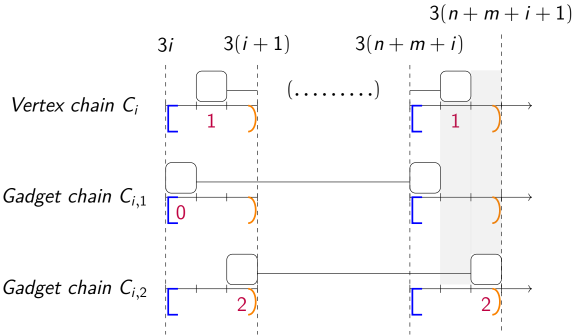
$1|chains(l^{min}_{i,j}), p_j = 1, r_j, d_j|\star$, parameter $\mu = 2$



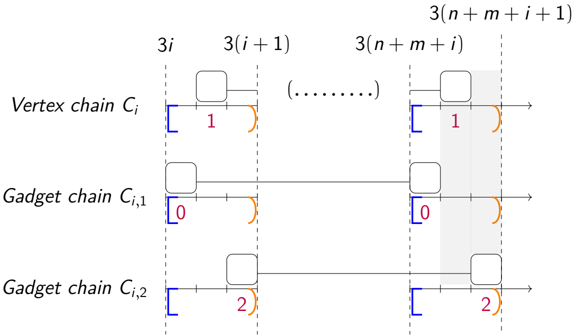
$1|chains(l_{i,j}^{min}), p_j = 1, r_j, d_j|\star$, parameter $\mu = 2$



$1|chains(l_{i,j}^{min}), p_j = 1, r_j, d_j|\star$, parameter $\mu = 2$



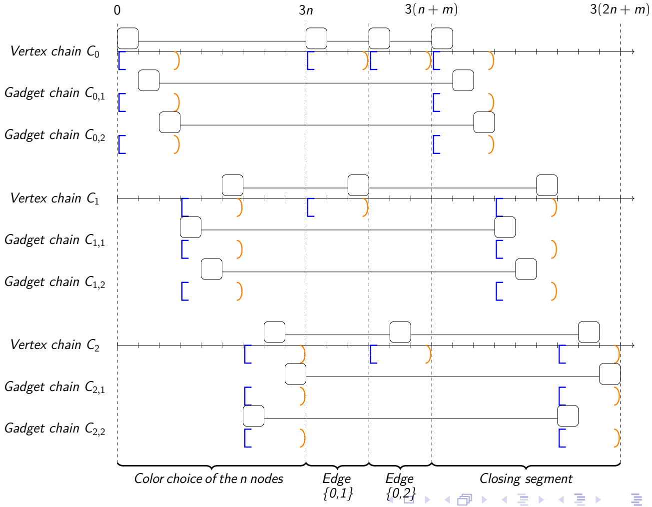
$1|chains(l_{i,j}^{min}), p_j = 1, r_j, d_j|\star$, parameter $\mu = 2$



$1|chains(l_{i,j}^{min}), p_j = 1, r_j, d_j|\star$, pathwidth 2 (cont.)

- Reduction from 3-COLORING: $G = (V, E)$, $n = |V|$, $m = |E|$

Example: $V = \{0, 1, 2\}$, $E = (\{0, 1\}, \{0, 2\})$



$1|chains(l_{i,j}^{min}), p_j = 1, r_j, d_j|\star$, pathwidth 2 (cont.)

- Reduction from 3-COLORING: $G = (V, E), n = |V|, m = |E|$

Example: $V = \{0, 1, 2\}, E = (\{0, 1\}, \{0, 2\})$