Parameterized complexity of a single machine scheduling problem

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FPT vs para-NP

- P $\subseteq$ NP
  - P: deterministic time $poly(n)$
  - NP: nondeterministic time $poly(n)$
- FPT $\subseteq$ para-NP, parameter k
  - FPT: deterministic time $f(k) \times poly(n)$
  - para-NP: nondeterministic time $f(k) \times poly(n)$
- P $=$ NP iff $\exists k$, FPT $=$ para-NP [Flum, Grohe 1998]

The W-hierarchy

- FPT $\subseteq$ W[1] $\subseteq$ W[2] $\subseteq \ldots \subseteq$ para-NP
  - W[1]-complete: INDEPENDENT SET, CLIQUE
  - W[2]-complete: DOMINATING SET

A few examples in scheduling

- [Bodlaender, Fellows 1995]
  - #machines: $P|prec, p_j = 1|C_{max}$ W[2]-hard
- [Mniech, Wiese 2015]
  - Maximum processing time p_{max}: $P||C_{max} \in$ FPT
- [Hermelin, Karhi, Pinedo, Shabtay 2018]
  - #distinct d_j, #distinct p_j, #distinct w_j: $1||\sum w_j U_j \in$ FPT with two of the three

Pathwidth μ

Definition: pathwidth

$$\mu \stackrel{\text{def}}{=} \max_{0 \le t < \max_j d_j} |\{j \mid r_j \le t < d_j\}| - 1$$

Recent work:

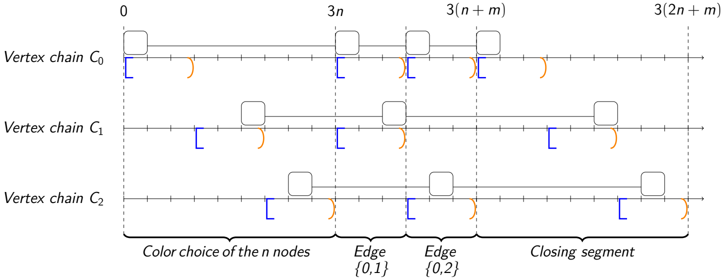
- $1|r_j, s_{jk}, reject, \tilde{d}_j| \sum_{j \notin R} v_j - \sum_{j \notin R} w_j T_j \in$ FPT (μ+slack) [de Weerdt et al. 2020]
- $P|prec, p_j = 1, r_j|L_{max} \in$ FPT (μ) [Munier-Kordon, Tang 2021]
- $P|M_j(type), prec, r_j|L_{max} \in$ FPT ($\mu + p_{max}$) [Hanen, Munier-Kordon 2022]

Problem definition

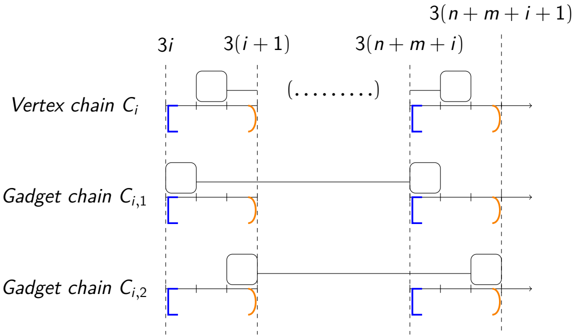
- Decision problem: $1|chains(l_{i,j}), p_j = 1, r_j, d_j|\star$
- Two cases:
  - EXACT DELAYS: $J_i \prec J_j \implies \sigma(j) = \sigma(i) + 1 + l_{i,j}$
  - MIN DELAYS: $J_i \prec J_j \implies \sigma(j) \geq \sigma(i) + 1 + l_{i,j}$
- Goal: prove para-NP-hardness in both cases
  - prove NP-hardness for a fixed value of the parameter [Flum, Grohe 1998]
    - Example: k-COLORING

- Reduction from 3-COLORING: $G = (V, E), n = |V|, m = |E|$

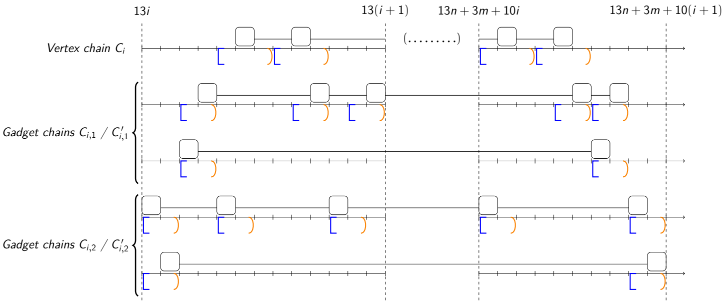
Example: $V = \{0, 1, 2\}, E = (\{0, 1\}, \{0, 2\})$



$1|chains(l_{i,j}), p_j = 1, r_j, d_j|\star$ MIN DELAYS, pathwidth 2



Improvement of MIN DELAYS with pathwidth 1



Conclusion

Theorem

$1|chains(l_{i,j}), p_j = 1, r_j, d_j|\star$ is para-NP-complete when parameterized by pathwidth μ in both EXACT DELAYS and MIN DELAYS cases.
More precisely this problem:

- is NP-hard when $\mu \geq 1$,
- can be solved in time $\mathcal{O}(n)$ when $\mu = 0$.

- Further work: investigate maximum delay l_{max}
  - FPT with parameter $\mu + l_{max}$?
  - hard with parameter l_{max} alone?

Thank you for your attention!